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Ergodic Control of Semilinear Stochastic Equations and the Hamilton–Jacobi Equation*

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In this paper we consider optimal control of stochastic semilinear equations with Lipschitz continuous drift and cylindrical noise. We show existence and uniqueness (up to an additive constant) of solutions to the stationary Hamilton–Jacobi equation associated with the cost functional given by the asymptotic average per unit time cost. As a consequence we find the optimizing controls given in the feedback form. To obtain these results we prove also some new results on the transition semigroups of semilinear diffusion acting in the spaces of continuous function with the weighted sup norms and on the optimal control of semilinear diffusions for the discounted cost functional. © 1999 Academic Press

Key Words: semilinear stochastic equation; invariant measure; Bismut–Elworthy formula; ergodic control; stationary Hamilton–Jacobi equation; dynamic programming.

1. INTRODUCTION

The aim of this paper is to study the ergodic control problem for a class of stochastic evolution equations of semilinear type. The basic example we

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have in mind is the following reaction-diffusion equation

$$\begin{cases} \dfrac{\partial X}{\partial t}(t,\zeta) = \dfrac{\partial^2 X}{\partial \zeta^2}(t,\zeta) + g(X(t,\zeta)) - u_t + \dfrac{\partial^2 W}{\partial t\, \partial \zeta}(t,\zeta), \\ X(t,0) = X(t,1) = 0, \\ X(0,\zeta) = x(\zeta), \qquad 0 \le \zeta \le 1, \qquad t \ge 0, \end{cases} \tag{1.1}$$

with the Dirichlet boundary conditions. The process W is a space-time white noise and g: $R \to R$ is a Lipschitz mapping. By u_t we denote the control which should be chosen so as to minimize the ergodic cost function

$$V(x,u) = \liminf_{T\to\infty} \left(\frac{1}{T} E_{x,u} \int_0^T \big(f(X(t,\cdot)) + h(u_t) \big)\, dt \right),$$

where $X(t,\cdot)$ is a solution to (1.1) in a suitable function space. The existence of optimal ergodic controls for stochastic semilinear equations has been studied in [18] and in [17] for the case of boundary controls. In this paper we show existence and uniqueness of solutions to the corresponding Hamilton–Jacobi equation (1.3) and thereby obtain the unique optimal control given in the feedback form. The only result of this type in the infinite dimensional framework we are aware of is [26], where the affine case is considered.

In the present work equation (1.1) is reformulated and generalized to a stochastic evolution equation

$$\begin{cases} dX_t = \big(AX_t + F(X_t) - u_t \big)\, dt + \sqrt{Q}\, dW_t, \\ X_0 = x \in X, \qquad t \ge 0, \end{cases} \tag{1.2}$$

in a separable Hilbert space $\mathscr{X}$ (for details concerning (1.2) see Section 2 below). In the case of one dimensional space $\mathscr{X}$ the ergodic control problem for (1.2) has been solved in [22] and later extended to any finite dimensions in [4] (see also [1, 3] and references therein). In order to solve the ergodic control problem for the case $\dim \mathscr{X} = \infty$ we will take a classical approach exploited in the aforementioned papers. Namely, we will use the Dynamic Programming Principle combined with the recent results on transition semigroups of infinite dimensional diffusions to show that there exists a unique (up to an additive constant) smooth solution (v, λ) to the associated Hamilton–Jacobi equation

$$\frac{1}{2}\operatorname{tr}\big(QD^2 v(x)\big) + \langle Ax + F(x), Dv(x)\rangle + f(x) - H(Dv(x)) - \lambda = 0, \tag{1.3}$$

where H denotes the Hamiltonian of the problem and D stands for the Fréchet derivative. Then λ gives the optimal cost of the ergodic control problem and $DH(Dv(\cdot))$ is the unique optimal feedback control.

In the finite dimensional case methods of solving (1.3) are relatively well developed. Usually Eq. (1.3) is approximated by a sequence of the stationary Hamilton–Jacobi equations

$$\alpha v_\alpha = \frac{1}{2}\operatorname{tr}\big(QD^2 v_\alpha(x)\big) + \langle Ax + F(x), Dv_\alpha(x)\rangle + f(x) - H\big(Dv_\alpha(x)\big) \tag{1.4}$$

on bounded domains. Then by the use of appropriate Sobolev-type estimates and compact imbeddings (see [2, 3] and references therein) it is shown that for α tending to zero $\alpha v_\alpha \to \lambda$ and $v_\alpha - \alpha^{-1}\langle v_\alpha, 1\rangle$ converges to the solution v of (1.3), where $\langle v_\alpha, 1\rangle$ is the average with respect to an appropriate invariant measure. For example, in the important paper [2] the problem in the whole space is approximated by a sequence of Neumann problems on bounded domains for which, due to the local Sobolev imbedding theorems, uniform estimates can be found. Unfortunately, results of this type are not available in infinite dimensions. Another approach is to use the powerful theory of viscosity solutions. Again, in infinite dimension serious problems arise if we want to apply the method of viscosity solutions to the case when the noise in Eq. (1.2) is cylindrical, or equivalently, when the operator Q in (1.3) has infinite trace. Recently a rapid progress has been made in this direction, see [25] and [20], mainly for the case of commuting A and Q with the discrete spectra. This method, however, does not yield differentiability of solution and the construction of optimal control needs additional work.

In this paper we take an approach initiated in [5], where the method of mild solution has been used to prove existence and uniqueness of the C^1 solution to a parabolic Hamilton–Jacobi equation. Later this method has been extended to the stationary case (see [19] and references therein). Our point of departure is the paper [19], where the solution (1.4) is defined as solution to the integral equation

$$v_\alpha(x) = \int_0^\infty e^{-\alpha s} R_s\big(f + \langle F, Dv_\alpha\rangle - H(Dv_\alpha)\big)(x)\, ds$$

and, if exists, is by definition a Fréchet differentiable function. In the above formula $R_s\phi(x) = E\phi(Z_s(x))$ stands for the transition semigroup of the Ornstein–Uhlenbeck process Z defined as

$$Z_t(x) = S(t)x + \int_0^t S(t-s)\sqrt{Q}\, dW_s.$$

In [19] the existence and uniqueness of solutions to (1.4) is proved under the assumption that f and F are bounded and uniformly continuous and F and H are Lipschitz mappings. It is shown also that v_α may be identified with the optimal cost for the problem of optimal control of (1.1) with discounted cost functional and then $DH(Dv_\alpha)$ provides the unique optimizing control. In our paper we prove similar results for the case of polynomially bounded f and the mapping F of linear growth. We show also that the optimal control is stabilizing and the closed loop equation has a unique invariant measure. We require, however, that $S(t)$ is a stable semigroup of contractions. F is dissipative and Gateaux differentiable, and Q is boundedly invertible. Moreover, we need to assume that controls take values in a sufficiently small bounded set. Note that in Eq. (1.1) A and Q satisfy the required conditions. An immediate consequence of these assumptions is that the process (Z_t) has a unique invariant measure μ and the rate of convergence to invariant measure is exponential. Another consequence is that the Bismut–Elworthy formula in the version proved in [14] holds for the transition semigroup of the semilinear equation

$$dY_t = \big(AY_t + F(Y_t)\big)\,dt + \sqrt{Q}\,dW_t. \tag{1.5}$$

Next, using the ergodic properties of the Ornstein–Uhlenbeck semigroup R_t, dissipativity of F and the Bismut–Elworthy formula we show that $\sup_{\alpha>0}\|Dv_\alpha\| < \infty$. Then the compactness result applied in the space $L^2(H,\mu)$ allows us to show that Eq. (1.3) has a solution. Finally, the Dynamic Programming Principle combined again with the ergodic properties of the considered diffusion processes yields uniqueness of solutions to (1.3).

In this paper the Hamilton–Jacobi equations (1.3) and (1.4) are considered in the spaces of uniformly continuous and polynomially increasing functions. However, an important ingredient of our proofs are the properties of the Ornstein–Uhlenbeck semigroup $R_t\phi(x) = E(Z_t(x))$ in the Sobolev spaces built on the invariant measure of the process $(Z_t(x))$. Let us mention here the recent work [13], where the Hamilton–Jacobi equation (1.3) is studied solely in the Gauss–Sobolev spaces.

Let us describe briefly the content of the paper. In Section 2 we present some auxiliary results. In Sections 3 and 4 we discuss the properties of the Ornstein–Uhlenbeck semigroup and the transition semigroup of the semilinear equation in the spaces of polynomially bounded and uniformly continuous functions. Some results of these section related to the characterization of generators of these semigroups are of independent interest. In Section 5 we prove the existence and uniqueness of solutions to (1.4) and in Section 6 the existence and uniqueness of solutions to (1.3).

2. FORMULATION OF THE PROBLEM AND AUXILIARY RESULTS

Consider an equation

$$\begin{aligned} dX_t &= \left(AX_t + F(X_t) - u_t\right) dt + \sqrt{Q}\, dW_t, \\ X_0 &= x \in X, \qquad t \geq 0, \end{aligned} \tag{2.1}$$

in a separable Hilbert space $\mathscr{X}$, where A: $\mathrm{dom}(A) \to \mathscr{X}$ generates a strongly continuous semigroup $S(\cdot)$ on $\mathscr{X}$, (W_t) is a cylindrical Wiener process on $\mathscr{X}$ defined on a probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. The operator Q is bounded and symmetric on $\mathscr{X}$ and $Q \geq 0$. Conditions (A1)–(A4) given below are always assumed to be satisfied throughout the paper and the results will be enunciated without recalling them.

(A1) The operator Q is boundedly invertible, and for a certain $\omega > 0$

$$\|S(t)\| \leq e^{-\omega t}.$$

Moreover, for a certain $T > 0$ and $\gamma > 0$

$$\int_0^T t^{-\gamma} \|S(t)\|_{\mathrm{HS}}^2\, dt < \infty,$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of an operator.

(A2) The mapping F is Lipschitz continuous and Gateaux differentiable. Moreover, there exists $\beta \in R$ such that

$$\langle DF(x)y, y\rangle \leq \beta \|y\|^2, \qquad x, y \in \mathscr{X}.$$

The control u is an $\mathscr{X}$-valued progressively measurable process. It is said to be admissible if $\sup_{s \geq 0} \|u_s\|^2 < \infty$. We say that $u \in \mathscr{U}_r$ for a certain $r > 0$ if u is admissible and $\sup_{s \geq 0} \|u_s\|^2 \leq r$. It is convenient to choose control in the feedback form, that is, $u_t = \tilde{u}(X_t)$, where

$$\tilde{u} \in \tilde{\mathscr{U}}_r = \{g : \mathscr{X} \to \mathscr{X} : \|g(x)\| \leq r \text{ for all } x \in \mathscr{X},\ g \text{ Borel measurable}\}.$$

In this case (2.1) takes the form

$$\begin{aligned} dX_t &= \left(AX_t + F(X_t) - \tilde{u}(X_t)\right) dt + \sqrt{Q}\, dW_t, \\ X_0 &= x \in \mathscr{X}, \qquad t \geq 0. \end{aligned} \tag{2.2}$$

In further considerations r will be fixed and for simplicity of notation we will write $\mathscr{U}$ and $\tilde{\mathscr{U}}$ instead of $\mathscr{U}_r$ and $\tilde{\mathscr{U}}_r$, respectively.

The main problem we are going to deal with is to find a control $\hat{u} \in \tilde{\mathscr{U}}$ such that

$$V(x, \hat{u}) = \inf_{u \in \mathscr{U}} V(x, u),$$

where

$$V(x, u) = \liminf_{T \to \infty} \frac{1}{T} E_{x,u} \int_0^T \big(f(X_t) + h(u_t)\big)\, dt \tag{2.3}$$

is the ergodic cost functional. In order to determine $\hat{u}$ we will need to solve the discounted cost problem: for every $\alpha > 0$ find $\hat{u}_\alpha \in \tilde{\mathscr{U}}$ such that

$$V_\alpha(x, \hat{u}_\alpha) = \inf_{u \in \mathscr{U}} V_\alpha(x, u),$$

where

$$V_\alpha(x, u) = E_{x,u} \int_0^\infty e^{-\alpha t}\big(f(X_t) + h(u_t)\big)\, dt. \tag{2.4}$$

We assume the following conditions.

(A3) There exists $m_0 \geq 0$ such that the function $f\colon \mathscr{X} \to R$ enjoys the property

$$\sup_{x \in \mathscr{X}} \frac{|f(x)|}{1 + \|x\|^{2m_0}} < \infty.$$

Moreover, the function

$$x \to \frac{f(x)}{1 + \|x\|^{2m_0}}$$

is uniformly continuous.

(A4) The function $h\colon B_r \to R_+$ is convex, lower semicontinuous and bounded, where

$$B_r = \{x \in \mathscr{X}\colon \|x\| \leq r\}, \qquad r > 0.$$

Moreover, the Hamiltonian of the problem given by the formula

$$H(x) = \sup_{\|y\| \leq r} \big(\langle y, x\rangle - h(y)\big)$$

is continuously differentiable.

We say that a predictable process (X_t) is a solution to (2.1) if for all $t \geq 0$

$$E_{x,u} \int_0^t \|X_s\|\, ds < \infty$$

and

$$X_t = S(t)x + \int_0^t S(t-s)(F(X_s) - u_s)\,ds + \int_0^t S(t-s)\sqrt{Q}\,dW_s.$$

For details see [15]. If there exists a filtered probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{P})$ and a cylindrical Wiener process $(\tilde{W}_t)$ on it such that the predictable process (X_t) defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{P})$ satisfies the equation

$$X_t = S(t)x + \int_0^t S(t-s)(F(X_s) - \tilde{u}(X_s))\,ds + \int_0^t S(t-s)\sqrt{Q}\,d\tilde{W}_s$$

then we say that Eq. (2.2) has a weak solution.

PROPOSITION 2.1. *For every $u \in \mathscr{U}$ Eq.* (2.1) *has a unique continuous solution X. Furthermore, Eq.* (2.2) *has a unique weak solution for each $\tilde{u} \in \tilde{\mathscr{U}}$. The solution to* (2.2) *induces a Markov Feller process in $\mathscr{X}$.*

Proof. If $u_t \equiv 0$ then the proposition follows from Theorem 7.13 in [15]. The proof for (2.1) is an easy modification. Let Y be a unique mild solution of the equation

$$\begin{aligned} dX_t &= (AX_t + F(X_t))\,dt + \sqrt{Q}\,dW_t, \\ X_0 &= x \in \mathscr{X}, \qquad t \geq 0. \end{aligned}$$

Then the weak solution for (2.2) is obtained by an absolutely continuous change of measure applied to the solution Y of (2.2). The change of measure is defined by the Girsanov exponential

$$\rho_T = \exp\left(-\int_0^T \left\langle Q^{-1/2}\tilde{u}(Y_s), dW_s \right\rangle - \frac{1}{2}\int_0^T \left\| Q^{-1/2}\tilde{u}(Y_s) \right\|^2 ds \right).$$

∎

LEMMA 2.2. *Assume that $\omega > \beta$. Then for each $p \geq 0$ there exist positive constants $C_1(p)$ and $C_2(p)$ such that for all $x \in \mathscr{X}$*

$$\sup_{t>0} \sup_{u \in \mathscr{U}} E_{x,u} \|X_t\|^{2p} \leq C_1(p)(1 + \|x\|^{2p}) \tag{2.5}$$

and for every $T > 0$

$$\sup_{u \in \mathscr{U}} E_{x,u} \sup_{t \leq T} \|X_t\|^{2p} \leq C_2(p)(1 + \|x\|^{2p}). \tag{2.6}$$

Proof. Without loss of generality we may assume that F is dissipative, that is, $\beta \leq 0$. Indeed, if $\beta > 0$ then we may replace A and F with $A + \beta$ and $F - \beta$, respectively. For $\phi \in L^1_{\mathrm{loc}}(R_+, \mathscr{X})$ let $y(\cdot, \phi)$ be a solution to the equation

$$y(t, \phi) = S(t)x + \int_0^t S(t-s)F(\phi + y(s, \phi))\, ds, \qquad t \geq 0, \quad (2.7)$$

and let

$$\tilde{Z}(t) = -\int_0^t S(t-s)u_s\, ds + \int_0^t S(t-s)\sqrt{Q}\, dW_s.$$

In view of (A1) standard estimates for stochastic integrals yield

$$\sup_{t>0} E\left\| \tilde{Z}(t) \right\|^{2p} < \infty, \qquad p \geq 0.$$

Hence $X_t = y(t, \tilde{Z}(\cdot)) + \tilde{Z}(t)$ and therefore it suffices to show that for certain $C_1, C_2 > 0$

$$\sup_{t>0} E\left\| y\big(t, \tilde{Z}(\cdot)\big) \right\|^{2p} \leq C_1 \|x\|^{2p} + C_2. \quad (2.8)$$

To this end we first use the Yosida approximations of F and then, proceeding in a similar way as in the proof of existence of solutions to (2.7) (cf. [15]), we obtain

$$\| y(t, \phi) \|^{2p} \leq e^{-2p\omega t} \|x\|^{2p} + \int_0^t e^{-2p\omega(t-s)} 2p \| F(\phi(s)) \| \| y(s, \phi) \|^{2p-1}\, ds. \quad (2.9)$$

Now, assume that inequality

$$\sup_{t>0} E\left\| y(t, \tilde{Z}) \right\|^{2p-1/2} \leq C_3 \|x\|^{2p-1/2} + C_4 \quad (2.10)$$

holds for some $C_3, C_4 > 0$. For

$$q = \frac{2p - 1/2}{2p - 1}, \qquad q' = \frac{q}{q-1}$$

we have by (2.10)

$$E\left\| y(t,\tilde{Z})\right\|^{2p} \le e^{-2p\omega t}\|x\|^{2p} + C_5\int_0^t e^{-2p\omega(t-s)}\left(\left\|F\left(\tilde{Z}(s)\right)\right\|^{q'}\right)^{1/q'} \times\left(E\left\| y(s,\tilde{Z})\right\|^{2p-1/2}\right)^{1/q} ds$$

$$\le C_3\|x\|^{2p} + C_4$$

Thus (2.5) is proved by induction. To show (2.6), note that (2.9) implies

$$\sup_{t\le T}\| y(t,\phi)\|^{2p} \le k_1\|x\|^{2p} + k_2\sup_{t\le T}\left(\| F(\phi(t))\|\| y(t,\phi)\|^{2p-1}\right);$$

thus

$$E_{x,u}\sup_{t\le T}\|X_t\|^{2p} \le E_{x,u}\left(k_3 + \sup_{t\le T}\left\| y(t,\tilde{Z})\right\|\right)$$

$$\le k_3 + k_1\|x\|^{2p} + k_4\left(E\sup_{t\le T}\left\|F\left(\tilde{Z}(t)\right)\right\|^{q'}\right)^{1/q'} \times\left(E\sup_{t\le T}\left\| y(t,\tilde{Z})\right\|^{2p-1/2}\right)^{1/q},$$

where q, q' are the same as above. The proof of (2.6) is again completed by induction. ∎

We are going to work in weighted spaces of continuous function which are defined as follows. Let

$$\rho_m(x) = \frac{2}{1+\|x\|^{2m}}, \qquad m = 0,1,\dots$$

and let $\mathscr{Y}$ be a Banach space. For a function $\phi\colon \mathscr{X}\to\mathscr{Y}$ we define the norm

$$\|\phi\|_m = \sup_{x\in\mathscr{X}}\left(\rho_m(x)\|\phi(x)\|\right).$$

The space $BUC_m(\mathscr{X},\mathscr{Y})$ is defined as the Banach space of functions $\phi\colon \mathscr{X}\to\mathscr{Y}$ such that $\|\phi\|_m < \infty$ and $\rho_m\phi$ is uniformly continuous. If $\mathscr{Y} = R$ then we write $BUC_m(\mathscr{X})$ and if $m = 0$ we write $BUC(\mathscr{X})$. If $\phi\colon \mathscr{X}\to\mathscr{Y}$ is k times Fréchet differentiable with $\rho_m D^k\phi$ uniformly continuous then we define

$$\|\phi\|_{k,m} = \|\phi\|_m + \|D\phi\|_m + \cdots + \|D^k\phi\|_m$$

and $BUC_m^k(\mathscr{X}) = \{\phi \in BUC_m(\mathscr{X})\colon \ \|\phi\|_{k,m} < \infty\}$. If $T\colon \ BUC_m(\mathscr{X}) \to BUC_m(\mathscr{X})$ is a bounded operator then its norm is denoted by $\|T\|_m$. Let

$$\|\phi\|_\theta = \|\phi\|_0 + \sup_{x \neq y \in \mathscr{X}} \frac{|\phi(x) - \phi(y)|}{\|x - y\|^\theta}, \qquad \theta \in (0,1),$$

denote the Hölder norm of the function ϕ in $BUC(\mathscr{X})$. The space

$$C^\theta(\mathscr{X}) = \{\phi \in BUC(\mathscr{X}) : \|\phi\|_\theta < \infty\},$$

endowed with the norm $\|\cdot\|_\theta$, is a Banach space continuously imbedded into $BUC(\mathscr{X})$. It has been shown in [6] that $(BUC(\mathscr{X}), BUC^1(\mathscr{X}))_{\theta,\infty} = C^\theta(\mathscr{X})$, where $(BUC(\mathscr{X}), BUC^1(\mathscr{X}))_{\theta,\infty}$ is a real interpolation space (for definition and basic properties see [6] and references therein). It is easy to see that the operator

$$T\phi = \frac{1}{\rho_m}\phi, \qquad \phi \in BUC(\mathscr{X}),$$

defines an isometry of $BUC(\mathscr{X})$ onto $BUC_m(\mathscr{X})$ and an isomorphism of $BUC^1(\mathscr{X})$ onto $BUC_m^1(\mathscr{X})$. Let

$$C_m^\theta(\mathscr{X}) = T\big(C^\theta(\mathscr{X})\big).$$

LEMMA 2.3. *For every* $\theta \in (0,1)$ *the space* $C_m^\theta(\mathscr{X})$ *is isomorphic to* $(BUC_m(\mathscr{X}), BUC_m^1(\mathscr{X}))_{\theta,\infty}$.

Proof. The lemma follows from [6] and the definition of the space $(BUC_m(\mathscr{X}), BUC_m^1(\mathscr{X}))_{\theta,\infty}$. ∎

Let μ be a centered Gaussian measure on $\mathscr{X}$ with the covariance operator C. The norm in the space $L^2(\mathscr{X}, \mu)$ will be denoted by

$$\|\phi\|_\mu^2 = \int_{\mathscr{X}} |\phi(x)|^2 \mu(dx).$$

Clearly $BUC_m^k(\mathscr{X}) \subset L^2(\mathscr{X}, \mu)$ with the continuous and dense embedding. For $\phi \in BUC^1(\mathscr{X})$ we define the norm

$$\|\phi\|_{1,\mu}^2 = \|\phi\|_\mu^2 + \int_{\mathscr{X}} \|D\phi(x)\|^2 \mu(dx).$$

The closure of $BUC^1(\mathscr{X})$ in the norm $\|\cdot\|_\mu$ is a subspace of $L^2(\mathscr{X}, \mu)$ and will be denoted by $W^{1,2}(\mathscr{X}, \mu)$.

We finish this section with a version of the Gronwall-type inequality given in [21].

LEMMA 2.4. *Assume that for $t > 0$*

$$\xi(t) \leq c_1 e^{-at} t^{-\beta} + c_2 \int_0^t e^{-a(t-s)} (t-s)^{-\beta} \xi(s)\, ds, \qquad (2.11)$$

where $\xi \geq 0$ is locally integrable, $\beta \in (0,1)$ and $a, c_1, c_2 > 0$. If $a > (c_2 \Gamma(\beta))^{1/\beta}$, where Γ denotes the Euler Gamma function, then

$$\xi(t) \leq c_1 \gamma(t) \qquad t\text{-a.e.}$$

for a certain function $\gamma \in L^1(0, \infty)$.

Proof. Set $u(t) = e^{at}\xi(t)$. By (2.11)

$$u(t) \leq \frac{c_1}{t^\beta} + c_2 \int_0^t \frac{\xi(s)}{(t-s)^\beta}\, ds, \qquad t > 0.$$

By Lemma 7.1.1 in [21], for any θ, $0 < \theta < (c_2 \Gamma(\beta))^{1/\beta}$, there exists a constant $c_0 > 0$ such that

$$u(t) \leq c_1 \left(\frac{1}{t^\beta} + c_0 \int_0^t (t-s)^{-\beta} e^{\theta(t-s)} s^{-\beta}\, ds \right).$$

Therefore $u(t) \leq c_1 \gamma(t)$ with

$$\gamma(t) = c_0 \left(e^{-at} t^{-\beta} + c_0 \int_0^t e^{(\theta - a)(t-s)} (t-s)^{-\beta} e^{-as} s^{-\beta}\, ds \right)$$

and the Young inequality yields $\gamma \in L^1(0, \infty)$. ∎

3. ORNSTEIN–UHLENBECK SEMIGROUP

In this section we consider a linear equation

$$\begin{aligned} dZ_t &= AZ_t\, dt + \sqrt{Q}\, dW_t \\ Z_0 &= x, \qquad t \geq 0. \end{aligned} \qquad (3.1)$$

We recall below basic properties of solutions to (3.1) and its transition semigroup; for details see [15]. The solution to (3.1) is given by the formula

$$Z_t = S(t)x + \int_0^t S(t-s)\sqrt{Q}\, dW_s.$$

Let

$$Q_t = \int_0^t S(t) Q S^*(t)\, dt, \qquad t \le \infty.$$

If (A1) holds then Q_t is well defined and nuclear for all $t \le \infty$; hence the process (Z_t) is well defined and Gaussian and for every $t \ge 0$ and $x \in \mathscr{X}$ the random variable Z_t has Gaussian distribution $N(S(t)x, Q_t)$. For a bounded Borel function $\phi\colon \mathscr{X} \to R$ we define the transition semigroup of the Ornstein–Uhlenbeck process

$$R_t \phi(x) = E_x \phi(Z_t).$$

Condition (A1) implies that the operator $\Gamma(t) = Q_t^{-1/2} S(t)$ is bounded on $\mathscr{X}$ and for a certain $c > 0$

$$\|\Gamma(t)\| < \frac{c}{\sqrt{t}}, \qquad t \in (0,1]. \tag{3.2}$$

It follows that the function $R_t \phi$ is Fréchet differentiable on $\mathscr{X}$,

$$\langle DR_t \phi(x), y \rangle = \int_{\mathscr{X}} \langle \Gamma(t) y, Q_t^{-1/2} z \rangle \phi(S(t)x + z)\, \mu_t(dz)$$

for all $x, y \in \mathscr{X}$ and

$$\|DR_t \phi(x)\| \le \|\Gamma(t)\| \sup_{z \in \mathscr{X}} |\phi(z)|.$$

Moreover, the centered Gaussian measure μ with the covariance operator Q_∞ is a unique and nondegenerate invariant measure for the semigroup (R_t). Finally, the semigroup (R_t) is a strongly continuous semigroup of contractions in the space $L^2(\mathscr{X}, \mu)$. Let

$$L_0^2(\mathscr{X}, \mu) = \{\phi \in L^2(\mathscr{X}, \mu) : \langle \phi, 1 \rangle_\mu = 0\}.$$

LEMMA 3.1. *Assume* (*A*1). *Then the following holds.*

(a) *For* $\epsilon \in (0,1)$

$$\int_0^\infty \|\Gamma(t)\|^{1+\epsilon}\, dt < \infty.$$

(b) *For every* $\alpha \ge 0$ *the operator*

$$T = \int_0^\infty e^{-\alpha t} R_t\, dt$$

is well defined and compact in $L_0^2(\mathscr{X}, \mu)$.

(c) *For every* $\alpha \geq 0$ *the operator*

$$DT = \int_0^\infty e^{-\alpha t} DR_t \, dt$$

is well defined and compact from $L^2(\mathscr{X}, \mu)$ *into* $L^2(\mathscr{X}, \mu; \mathscr{X})$.

Proof. (a) Note first that by (A1)

$$\operatorname{im}\left(Q_t^{1/2}\right) = \operatorname{im}\left(Q_\infty^{1/2}\right), \qquad t > 0.$$

Hence $\Gamma(t) = B(t) Q_\infty^{-1/2} S(t)$ with the operator $B(t) = Q_t^{-1/2} Q_\infty^{1/2}$ bounded and using the notation $|T| = (T^* T)^{1/2}$ we find that $\|\Gamma(t)\| = \| \, |B^*(t)| Q_\infty^{-1/2} S(t) \|$. By the result in [11] $|B^*(t)| = (I - S_0(t) S_0^*(t))^{-1/2}$, where $S_0(t)$ is a C_0-semigroup on $\mathscr{X}$ enjoying the property $\|S_0(t)\| \leq e^{-\lambda t}$ for a certain $\lambda > 0$. Moreover, the function $t \to \langle S_0(t) S_0^*(t) x, x \rangle$ is decreasing for every $x \in \mathscr{X}$. Hence, taking (3.2) into account

$$\begin{aligned}
\int_0^\infty \|\Gamma(t)\|^{1+\varepsilon} dt &= \sum_{k=0}^\infty \int_k^{k+1} \|\Gamma(t)\|^{1+\epsilon} dt \\
&= \sum_{k=0}^\infty \int_0^1 \left\| |B^*(s+k)| Q_\infty^{-1/2} S(s+k) \right\|^{1+\epsilon} ds \\
&\leq \sum_{k=0}^\infty \int_0^1 \left\| |B^*(s)| Q_\infty^{-1/2} S(s) \right\|^{1+\epsilon} \|S(k)\|^{1+\epsilon} ds \\
&\leq \sum_{k=0}^\infty e^{-\omega(1+\epsilon)k} \int_0^1 \|\Gamma(s)\|^{1+\epsilon} ds \leq \frac{2c^{1+\epsilon}}{1-\epsilon} \frac{1}{1 - e^{-\omega(1+\epsilon)}}
\end{aligned}$$

and (a) follows.

(b) It is enough to consider the case $\alpha = 0$. By the result in [10] the semigroup (R_t) is compact in $L^2(\mathscr{X}, \mu)$ and by the result from [12]

$$\|R_t\|_{L_0^2 \to L_0^2} \leq e^{-\lambda t}.$$

Hence (b) follows easily.

(c) Again, we may assume $\alpha = 0$. By the result in [10] the operator DR_t: $L^2(\mathscr{X}, \mu) \to L^2(\mathscr{X}, \mu; \mathscr{X})$ is compact for every $t > 0$. Since $\|DR_t\|_{L^2 \to L^2} \leq \|\Gamma(t)\|$, (a) yields the desired result. ∎

We will consider now the semigroup (R_t) in the space $BUC_m(\mathscr{X})$ using the ideas introduced in [7, 8]. Note that (A1) yields

$$R_t\left(\frac{1}{\rho_m}\right)(x) = \frac{1}{2}\left(1 + \int_X \|S(t)x + y\|^{2m} \mu_t(dy)\right) \le \frac{b}{\rho_m(x)} \tag{3.3}$$

with $b = b(m, a, Q) > 0$ independent of t.

In the lemma below we collect some properties of the semigroup (R_t) in $BUC_m(\mathscr{X})$.

LEMMA 3.2. (a) *For every $t \ge 0$ R_t is a bounded operator on $BUC_m(\mathscr{X})$. Moreover, for some constants $d_1, d_2 > 0$*

$$M = \sup_{t \ge 0} \|R_t\|_m < \infty, \tag{3.4}$$

$$\|DR_t\phi\|_m \le d_1\|\Gamma(t)\| \, \|\phi\|_m, \qquad t > 0, \tag{3.5}$$

$$\|D^2R_t\phi\|_m \le d_2\|\Gamma(t)\|^2 \|\phi\|_m, \qquad t > 0. \tag{3.6}$$

(b) *For every $m \ge 0$ and $\theta \in (0,1)$ there exists $C(\theta) > 0$ such that*

$$\|DR_t\phi\|_{\theta, m} \le C(\theta) M_\theta(t) \|\phi\|_m \tag{3.7}$$

with $M_\theta(t) = (1 + \|\Gamma(t)\|)^\theta \|\Gamma(t)\|$. Moreover, $M_\theta \in L^1(0,\infty)$ for θ small enough.

(c) *Let $(\phi_n) \subset BUC_m(\mathscr{X})$ be a sequence converging to $\phi \in BUC_m(\mathscr{X})$ uniformly on compacts and such that $\sup_{n \ge 1}\|\phi_n\|_m < \infty$. Then $\sup_{n \ge 1}\|R_t\phi_n\|_m < \infty$. Moreover,*

$$\lim_{t \to 0} R_t\phi(x) = \phi(x) \quad \text{and} \quad \lim_{n \to \infty} R_t\phi_n(x) = R_t\phi(x)$$

uniformly on compacts.

Proof. Parts (a) and (c) were proved in [7]. Then (a) and Lemma 2.3 yield for every $m > 0$

$$\begin{aligned}
\|DR_t\phi\|_{\theta, m} &\le C(\theta)\|DR_t\phi\|_{1,m}^{\theta} \, \|DR_t\phi\|_m^{1-\theta} \\
&\le C(\theta)\left(\|\Gamma(t)\| + \|\Gamma(t)\|^2\right)^2 \|\phi\|_m^{\theta} \|\Gamma(t)\|^{1-\theta} \, \|\phi\|_m^{1-\theta} \\
&= C(\theta) M_\theta(t) \|\phi\|_m
\end{aligned}$$

and for θ small enough $M_\theta \in L^1(0,\infty)$ by Lemma 3.1. ∎

For $\phi \in BUC(\mathscr{X})$ and $\alpha > 0$ we define an operator

$$J_\alpha \phi(x) = \int_0^\infty e^{-\alpha t} R_t \phi(x)\, dt.$$

PROPOSITION 3.3. (a) *For every* $\alpha > 0$ *the operator* J_α *extends to a bounded operator on* $BUC_m(\mathscr{X})$ *and*

$$\|J_\alpha\|_m \le \frac{M}{\alpha}.$$

Moreover, for $\alpha, \beta > 0$ *the resolvent identity holds*:

$$J_\alpha - J_\beta = (\beta - \alpha) J_\beta J_\alpha.$$

(b) *There exists in* $BUC_m(\mathscr{X})$ *a closed operator* L_m *such that* $J_\alpha \phi = (\alpha - L_m)^{-1}\phi$ *for every* $\alpha > 0$ *and* $\phi \in BUC_m(\mathscr{X})$.

Proof. We have

$$\sup_{x \in \mathscr{X}} \left(\rho_m(x) |J_\alpha \phi(x)| \right) \le \sup_{x \in \mathscr{X}} \int_0^\infty e^{-\alpha t} \rho_m(x) |R_t \phi(x)|\, dt \le \frac{M}{\alpha} \|\phi\|_m$$

which proves the first estimate in (a). The resolvent identity follows from standard computation. The proof of (b) follows immediately from (a) and Theorem 1 of [27, p. 216]. ∎

The semigroup (R_t) is not strongly continuous in $BUC_m(\mathscr{X})$ and not even measurable. Hence, we cannot define the generator of this semigroup by the standard formula

$$L\phi = \lim_{t \to 0} \frac{R_t \phi - \phi}{t},$$

where the convergence is understood in the sense of the norm topology on $BUC_m(\mathscr{X})$ (see, however, Lemma 3.5 below). Instead, we will follow [7] and use the resolvent of the semigroup (R_t) in order to define the generator.

DEFINITION 3.4. The operator L_m will be called a generator of the semigroup (R_t) acting in the space $BUC_m(\mathscr{X})$.

Following [9] we say that a sequence $(\phi_n) \subset BUC_m(\mathscr{X}, \mathscr{Y})$ is $\mathscr{K}$-convergent to ϕ if

$$\sup_{n \ge 1} \|\phi_n\|_m < \infty \quad \text{and} \quad \lim_{n \to \infty} \phi_n(x) = \phi(x)$$

uniformly on compact subsets of $\mathscr{X}$. Hence, Lemma 3.2(b) says that for every $t \geq 0$ the operator R_t is $\mathscr{K}$-continuous on $BUC_m(\mathscr{X})$ and $R_t \phi$ is $\mathscr{K}$-convergent to ϕ for t tending to zero.

For $\phi \in BUC(\mathscr{X})$ we define the space

$$\mathscr{D}_0 = \Big\{ \phi \in BUC^2(\mathscr{X}) \colon \|\phi\|_0 + \big\| \operatorname{tr}\big(D^2\phi\big)\big\|_0 + \|D\phi\|_0 + \big\| \langle \,\cdot\,, A^* D\phi \rangle \big\|_0 < \infty \Big\}$$

and the operator

$$\mathscr{A}\phi(x) = \frac{1}{2}\operatorname{tr}\big(QD^2\phi(x)\big) + \langle Ax, D\phi(x) \rangle, \qquad \phi \in \mathscr{D}_0 .$$

If $\phi \in \mathscr{D}_0$ then $\mathscr{A}\phi \in BUC_m(\mathscr{X})$ for all $m = 0, 1, \dots$. The operator $\mathscr{A}$ with the domain $\mathscr{D}_0$ has been introduced in [9].

LEMMA 3.5. *The generator L_m of the semigroup (R_t) in $BUC_m(\mathscr{X})$ enjoys the following properties.*

(i) *If $\phi \in \operatorname{dom}(L_m)$ then for all $x \in \mathscr{X}$*

$$\lim_{t \to 0} \frac{R_t \phi(x) - \phi(x)}{t} = L_m \phi(x).$$

(ii) *The operator L_m is $\mathscr{K}$-closed and $\operatorname{dom}(L_m)$ is $\mathscr{K}$-dense in $BUC_m(\mathscr{X})$.*

(iii) *For every $\phi \in \operatorname{dom}(L_m)$ there exists a sequence $(\phi_n) \subset \mathscr{D}_0$ such that*

$$\phi_n \text{ is } \mathscr{K}\text{-convergent to } \phi,$$

$$D\phi_n \text{ is } \mathscr{K}\text{-convergent to } D\phi,$$

$$L_m \phi_n \text{ is } \mathscr{K}\text{-convergent to } L_m \phi$$

in $BUC_m(\mathscr{X})$ for every $m = 0, 1, \dots$. Moreover, $(L_m, \operatorname{dom}(L_m))$ is a $\mathscr{K}$-closure of $(\mathscr{A}, \mathscr{D}_0)$ for every $m = 0, 1, \dots$

Proof. The proof is a modification of the proof of an analogous statement for $m = 0$ given in [9] and is omitted. ∎

4. THE TRANSITION SEMIGROUP OF SEMILINEAR EQUATION

In this section we consider an uncontrolled version of (2.1):

$$\begin{aligned} dX_t &= \big(AX_t + F(X_t)\big)\, dt + \sqrt{Q}\, dW_t, \\ X_0 &= x \in \mathscr{X}, \qquad t \geq 0. \end{aligned} \tag{4.1}$$

For $\phi \in BUC_m(\mathscr{X})$ the transition semigroup $P_t\phi(x) = E\phi(X_t^x)$ of the process X is well defined due to Lemma 2.2. The main tool in further analysis is a version of the Bismut–Elworthy formula proved in [14]. If (A1) and (A2) hold then for all $x, y \in \mathscr{X}$ and $\phi \in BUC(\mathscr{X})$

$$\langle DP_t\phi(x), y\rangle = \frac{1}{t}E\left(\phi(X_t^x)\int_0^t \langle Q^{-1/2}Y_s^{x,y}, dW_s\rangle\right), \tag{4.2}$$

where $Y_t^{x,y}$ is the unique solution of the Cauchy problem

$$\begin{aligned} \dot{Y}_t^{x,y} &= \left(A + DF(X_t^x)\right)Y_t^{x,y}, \\ Y_0^{x,y} &= y, \qquad t \geq 0. \end{aligned} \tag{4.3}$$

The process $(Y_t^{x,y})$ may be identified with the process of mean square Gateaux derivatives of the process (X_t^x) with respect to x in the direction y.

PROPOSITION 4.1. *Assume that $\omega > \beta$. Then for every $m \geq 0$ the following holds.*

(i)

$$M_m = \sup_{t\geq 0}\|P_t\|_m \leq 1 + C_1(m),$$

where the constant $C_1(m)$ is defined in Lemma 2.2.

(ii) *For each $\phi \in BUC_m(\mathscr{X})$ and $t > 0$ the function $P_t\phi$ is Fréchet differentiable, formula* (4.2) *holds, and*

$$\sup_{x\in\mathscr{X}}\left(\rho_m(x)\|DP_t\phi(x)\|\right) \leq \frac{C}{\sqrt{t}}\|\phi\|_m, \tag{4.4}$$

where

$$C = \begin{cases} \sqrt{\dfrac{1 + C_1(2m)}{\|Q\|}} & \text{if } m > 0, \\ \dfrac{1}{\sqrt{\|Q\|}} & \text{if } m = 0. \end{cases}$$

Moreover, for $t \geq 1$

$$\sup_{x\in\mathscr{X}}\left(\rho_m(x)\|DP_t\phi(x)\|\right) \leq Ce^{-(\omega-\beta)(t-1)}\|\phi\|_m. \tag{4.5}$$

In particular, for each $\omega_1 \in (0, \omega - \beta)$ *and* $t > 0$

$$\sup_{x \in \mathscr{X}} \left(\rho_m(x) \| DP_t \phi(x) \| \right) \leq k(\omega_1) \frac{C}{\sqrt{t}} e^{-\omega_1 t}, \tag{4.6}$$

where

$$k(\omega_1) = \max\left\{ \exp(\omega_1), \frac{\exp(\omega - \beta - (1/2))}{\sqrt{2(\omega - \beta - \omega_1)}} \right\}.$$

Proof. (i) Invoking Lemma 2.2 with $p = m$ and $u = 0$ we obtain

$$\sup_{x \in \mathscr{X}} \left(\rho_m(x) |P_t \phi(x)| \right) \leq \|\phi\|_m \sup_{x \in \mathscr{X}} \left(\rho_m(x) \frac{1}{2} E\left(1 + \|X_t^x\|^{2m}\right) \right)$$

and

$$E\left(1 + \|X_t^x\|^{2m}\right) \leq \left(1 + C_1(m)\right)\left(1 + \|x\|^{2m}\right).$$

Therefore, by definition of ρ_m

$$\sup_{x \in \mathscr{X}} \left(\rho_m(x) |P_t \phi(x)| \right) \leq \left(1 + C_1(m)\right) \|\phi\|_m$$

and (i) follows.

(ii) To prove (4.5) we will show first that

$$\|Y_t^{x,y}\| \leq \|y\| e^{-t(\omega - \beta)}, \qquad t \geq 0, \quad y \in \mathscr{X}. \tag{4.7}$$

To this end we approximate the solution to (4.3) by a sequence of strong solutions. Let $g_n(x) = n(n - A)^{-1} DF(x)$, $y_n = n(n - A)^{-1} y$ and let u_n be a unique strong solution of the equation

$$\begin{aligned} \dot{u}_n &= \left(A + g_n(X_t^x) \right) u_n, \\ u_n(0) &= y_n. \end{aligned} \tag{4.8}$$

Then

$$\begin{aligned} \frac{d}{dt} \|u_n(t)\|^2 &= 2\left\langle \left(A + g_n(X_t^x) \right) u_n(t), u_n(t) \right\rangle \\ &\leq 2\left(-\omega \|u_n(t)\|^2 + \left\langle g_n(X_t^x) u_n(t), u_n(t) \right\rangle \right) \end{aligned}$$

and thereby

$$\|u_n(t)\|^2 \le \|y_n\|^2 - \omega\int_0^t \|u_n(s)\|^2\,ds + \int_0^t \langle g_n(X_s^x)u_n(s), u_n(s)\rangle\,ds.$$

Standard arguments allow us to pass with n to infinity in the above inequality. Therefore, taking (A2) into account we obtain

$$\|Y_t^{x,y}\|^2 \le \|y\|^2 - 2(\omega - \beta)\int_0^t \|Y_s^{x,y}\|^2\,ds$$

and the Gronwall Lemma yields (4.7). Setting $\phi_n = \min(\phi, n)$ for $\phi \in BUC_m(\mathscr{X})$, we find that

$$\langle DP_t\phi_n(x), y\rangle = \frac{1}{t}E\left(\phi_n(X_t^x)\int_0^t \langle Q^{-1/2}Y_s^{x,y}, dW_s\rangle\right) \tag{4.9}$$

and

$$\begin{aligned}|\langle DP_t\phi_n(x), y\rangle| &\le \frac{1}{t}\left(\sqrt{E\phi_n^2(X_t^x)}\right)\sqrt{E\int_0^t\|Q^{-1/2}Y_s^{x,y}\|^2\,ds}\\ &\le \frac{1}{t}\sqrt{E\left(\frac{1}{\rho_m(X_t^x)}\right)^2}\,\|\phi_n\|_m\frac{\sqrt{t}}{\sqrt{\|Q\|}}\|y\|.\end{aligned}$$

This estimate yields immediately (4.4) for $m = 0$. To prove (4.4) for $m > 0$ note first that $\rho_m^{-2} \in BUC_{2m}(\mathscr{X})$ and thereby Lemma 4.1 yields

$$\begin{aligned}\sup_{x\in\mathscr{X}}\left(\rho_m(x)\|DP_t\phi_n\|\right) &\le \frac{1}{\sqrt{t}}\frac{1}{\sqrt{\|Q\|}}\|\phi_n\|_m\\ &\quad\times\sqrt{\sup_{x\in\mathscr{X}}\left(\frac{\rho_m^2(x)}{\rho_{2m}(x)}\right)(1 + C_1(2m))\|\rho_m^{-2}\|_{2m}}\\ &\le \frac{1}{\sqrt{t}}\frac{1}{\sqrt{\|Q\|}}\|\phi_n\|_m\sqrt{1 + C_1(2m)}\,.\end{aligned}$$

Therefore

$$\sup_{n\ge 1}\|DP_t\phi_n\|_m \le \frac{1}{\sqrt{t}}\frac{1}{\sqrt{\|Q\|}}\sqrt{1 + C_1(2m)}\,\sup_{n\ge 1}\|\phi_n\|_m < \infty$$

and the obtained estimates allow us to pass to infinity with n in the right hand side of (4.9). On the other hand

$$P_t \phi_n(x + \zeta y) - P_t \phi_n(x) = \int_0^{\zeta} \langle DP_t \phi_n(x + sy), y \rangle \, ds.$$

Applying (4.7) and taking the limit with respect to n we find that (4.2) holds for any $\phi \in BUC_m(\mathscr{X})$ and

$$\sup_{x \in \mathscr{X}} \left(\rho_m(x) \| DP_t \phi(x) \| \right) \leq \frac{1}{\sqrt{t}} \sqrt{\frac{1 + C_1(2m)}{\|Q\|}} \, \|\phi\|_m$$

which proves (4.4). Moreover, the function $\phi_1 = P_1 \phi$ is Fréchet differentiable and

$$\begin{aligned} \langle DP_t \phi(x), y \rangle &= \langle DP_{t-1} \phi_1(x), y \rangle = \langle D\left(E\phi_1(X_{t-1}^x)\right), y \rangle \\ &= E \langle D\phi_1(X_{t-1}^x), Y_{t-1}^{x,y} \rangle, \qquad t > 1, \qquad x, y \in \mathscr{X}. \end{aligned}$$

Therefore, (4.5) follows from (4.7) and (4.4). Finally, using (4.4) and (4.5) we obtain (4.6). ∎

PROPOSITION 4.2. *For every $m \geq 0$ the operator P_t is bounded on $BUC_m(\mathscr{X})$.*

Proof. By Proposition 4.1 it is enough to show that the function $\rho_m P_t \phi$ is uniformly continuous for every $\phi \in BUC_m(\mathscr{X})$. This follows easily from the second part of Proposition 4.1. ∎

Let (F_n) be a sequence of Fréchet differentiable mappings from $\mathscr{X}$ to $\mathscr{X}$ such that F_n is $\mathscr{K}$-convergent to F and DF_n is $\mathscr{K}$-convergent to DF. For every $n \geq 1$ we define the process X^n as a unique solution to the equation

$$\begin{aligned} dX_t^n &= \left(AX_t^n + F_n(X_t^n)\right) dt + \sqrt{Q} \, dW_t, \\ X_0^n &= x \in \mathscr{X}, \qquad t \geq 0. \end{aligned}$$

The corresponding transition semigroup is denoted by $P_t^n \phi(x) = E_x \phi(X_t^n)$.

LEMMA 4.3. *For every $\phi \in BUC_m(\mathscr{X})$ and $t > 0$*

$$\lim_{n \to \infty} P_t^n \phi(x) = P_t \phi(x) \quad \text{and} \quad \lim_{n \to \infty} DP_t^n \phi(x) = DP_t \phi(x)$$

uniformly on compact subsets of $\mathscr{X}$.

Proof. The proof follows from the definition of the semigroup (P_t^n), the weak convergence of the processes X^n to the process X, and the formula (4.2). ∎

LEMMA 4.4. *For every* $\phi \in BUC_m(\mathscr{X})$ *and* $x \in \mathscr{X}$

$$P_t\phi(x) = R_t\phi(x) + \int_0^t R_{t-s}(\langle F, DP_s\phi\rangle)(x)\,ds. \tag{4.10}$$

Moreover, if $\phi \in \operatorname{dom}(L_m)$ *then for every* $x \in \mathscr{X}$

$$\lim_{t\to 0} \frac{P_t\phi(x) - \phi(x)}{t} = L_m\phi(x) + \langle F(x), D\phi(x)\rangle. \tag{4.11}$$

Proof. Let (F_n) be a sequence which satisfies conditions of Lemma 4.3 and let P_t^n be the corresponding semigroup. Then by the result in [15]

$$P_t^n\phi(x) = R_t\phi(x) + \int_0^t R_{t-s}(\langle F_n, DP_s^n\phi\rangle)(x)\,ds, \qquad x \in \mathscr{X},$$

for every $\phi \in BUC(\mathscr{X})$. Moreover, by the result in [10]

$$R_t\phi(x) = \int_{\mathscr{X}} G(t,x,y)\phi(y)\mu(dy), \qquad \phi \in L^2(\mathscr{X},\mu),$$

where μ is the invariant measure of the semigroup (R_t) and G denotes its transition kernel. Hence for every $n \geq 1$

$$P_t^n\phi(x) = R_t\phi(x) + \int_0^t \int_{\mathscr{X}} G(t-s,x,y)\langle F_n(y), DP_s^n\phi(y)\rangle\mu(dy)\,ds.$$

Due to Lemma 4.3 we can pass with n to infinity and obtain (4.10) for $\phi \in BUC(\mathscr{X})$. For any $\phi \in BUC_m(\mathscr{X})$ the function $s \to \langle F(x), DP_s\phi(x)\rangle$ is continuous for $s > 0$ and integrable on $[0,t]$ by Proposition 4.1. Hence the operator

$$\phi \to \int_0^t R_{t-s}(\langle F, DP_s\phi\rangle)(\cdot)\,ds$$

is well defined on $BUC_m(\mathscr{X})$ and coincides with $P_t - R_t$ which conclude the proof of (4.10). Since $\operatorname{dom}(L_m) \subset BUC_m^1(\mathscr{X})$ Eq. (4.11) follow easily from (4.10). ∎

Similarly as for the Ornstein–Uhlenbeck transition semigroup we define the resolvent of the semigroup (P_t) pointwise by the formula

$$J_\alpha^F\phi(x) = \int_0^\infty e^{-\alpha t}P_t\phi(x)\,dt, \qquad \alpha > 0, \quad \phi \in BUC_m(\mathscr{X}), \quad x \in \mathscr{X}.$$

THEOREM 4.5. *For every $\alpha > 0$ the operator J_α^F is bounded in $BUC_m(\mathscr{X})$,*

$$\|J_\alpha^F \phi\|_m \leq \frac{1 + C_1(m)}{\alpha} \|\phi\|_m \tag{4.12}$$

and $J_\alpha^F - J_\beta^F = (\beta - \alpha) J_\alpha^F J_\beta^F$ for any $\alpha, \beta > 0$. Finally, for any $\alpha > 0$ and $\phi \in BUC_m(H)$

$$J_\alpha^F \phi(x) = J_\alpha \phi(x) + J_\alpha\big(\langle F, DJ_\alpha^F \phi\rangle\big)(x), \qquad x \in \mathscr{X}, \tag{4.13}$$

where we use the same notation J_α for the resolvent of the semigroup (R_t) in $BUC_m(\mathscr{X})$ and $BUC_{m+1}(\mathscr{X})$. In particular, if $\|D\phi\|_0 < \infty$ then $J_\alpha^F \phi \in \mathrm{dom}(L_m)$ for $m \geq 1$.

Proof. The estimate (4.2) follows from Lemma 4.1 and by standard arguments we obtain the resolvent identity for y_α^F. Using the same arguments as in the proof of Proposition 4.1 we find that for certain constants $b_1, b_2 > 0$

$$\left| \rho_m(x) J_\alpha^F \phi(x) - \rho_m(y) J_\alpha^F \phi(y) \right| \leq \left(b_1 + b_2 \int_0^\infty \frac{1}{\sqrt{t}} e^{-\alpha t}\, dt \right) \|\phi\|_m \|x - y\|,$$

which yields uniform continuity of $\rho_m J_\alpha^F \phi$ and the continuity of J_α^F on $BUC_m(\mathscr{X})$ follows. For every $\phi \in BUC_m(\mathscr{X})$ Lemma 4.4 yields

$$\begin{aligned} J_\alpha^F \phi(x) &= \int_0^\infty e^{-\alpha t} R_t \phi(x)\, dt \\ &\quad + \int_0^\infty e^{-\alpha s} \int_s^\infty e^{-\alpha(t-s)} R_{t-s}(\langle F, DP_s \phi\rangle)(x)\, dt\, ds \\ &= \int_0^\infty e^{-\alpha t} R_t \phi(x)\, dt + \int_0^\infty e^{-\alpha t} R_t \left(\left\langle F, D \int_0^\infty e^{-\alpha s} P_s \phi\, ds \right\rangle \right)(x)\, dt \\ &= \int_0^\infty e^{-\alpha t} R_t\big(\langle F, DJ_\alpha^F \phi\rangle + \phi\big)(x)\, dt \end{aligned}$$

as desired. The last part of the theorem follows immediately from (4.12). ∎

COROLLARY 4.6. (a) *For every $m \geq 0$ there exists a unique closed operator L_m^F in $BUC_m(\mathscr{X})$ such that $J_\alpha^F = (\alpha - L_m^F)^{-1}$ for every $\alpha > 0$.*

(b) *The operator L_m^F is $\mathscr{K}$-closed. Moreover,* $\mathrm{dom}(L_m^F) \subset \mathrm{dom}(L_{m+1}) \cap BUC_m(\mathscr{X})$ *for every $m \geq 0$ and for every $\phi \in \mathrm{dom}(L_m^F)$*

$$L_m^F \phi(x) = L_{m+1} \phi(x) + \langle F(x), D\phi(x)\rangle, \qquad x \in \mathscr{X}. \tag{4.14}$$

If $D\phi \in BUC_0(\mathscr{X})$ then $\phi \in \mathrm{dom}(L_m)$.

Proof. Since the operator J_α^F is bounded in $BUC_m(\mathscr{X})$ and satisfies the resolvent identity by Theorem 4.6, part (a) follows from Theorem 1 of [27, p. 216]. The proof that L_m^F is $\mathscr{K}$-closed follows by the same arguments as in the proof of Remark 2.11 in [9]. To prove the last part of the corollary take $\phi \in \operatorname{dom}(L_m^F)$. Then $\phi = J_\alpha^F \psi$ for a certain $\psi \in BUC_m(\mathscr{X})$. If

$$\sup_{x \in \mathscr{X}} \frac{(1 + \|x\|)\| D\psi(x)\|}{1 + \|x\|^{2m}} < \infty \tag{4.15}$$

then $\langle F, D\phi \rangle \in BUC_m(\mathscr{X})$ and by (4.13) and Lemma 4.4 (4.14) holds. Assume that $\psi \in BUC_m(\mathscr{X})$ is arbitrary. Then there exists a sequence (ψ_n) which is $\mathscr{K}$-convergent to ψ and (4.15) holds for every ψ_n. As a consequence we find that the sequence $\phi_n = J_\alpha^F \psi_n$ is $\mathscr{K}$-convergent to ϕ and

$$L_m^F \phi_n(x) = L_m \phi_n(x) + \langle F(x), D\phi_n(x) \rangle, \qquad x \in \mathscr{X}. \tag{4.16}$$

Moreover, $L_m^F \phi_n(x) = \alpha J_\alpha^F \psi_n(x) - \psi_n(x) \to L_m^F \phi(x)$ in the sense of $\mathscr{K}$-convergence since L_m^F is $\mathscr{K}$-closed and by the same argument $L_{m+1}^F \phi_n$ is $\mathscr{K}$-convergent to $L_{m+1}^F \phi = L_m^F \phi$. Because DJ_α^F is a bounded operator on $BUC_m(\mathscr{X})$ and $\mathscr{K}$-continuous it follows also that $DJ_\alpha^F \psi_n$ is $\mathscr{K}$-convergent to $DJ_\alpha^F \psi$ in $BUC_{m+1}(\mathscr{X})$. Therefore $L_{m+1}\phi_n$ is also $\mathscr{K}$-convergent and since L_{m+1} is closed we find that $\phi \in \operatorname{dom}(L_{m+1})$ and $L_{m+1}\phi_n$ is $\mathscr{K}$-convergent to $L_{m+1}\phi$. Finally, we can conclude the proof of (4.14) passing to the limit in (4.16). ∎

DEFINITION 4.7. The operator L_m^F defined in Corollary 4.6 will be called the generator of the transition semigroup (P_t) on $BUC_m(\mathscr{X})$.

In Section 5 we will need properties of the transition semigroup corresponding to the equation

$$\begin{aligned} dX_t^x &= \left(AX_t^x + F(X_t^x) - \tilde{u}(X_t^x)\right) dt + \sqrt{Q}\, dW_t, \\ X_0^x &= x, \end{aligned} \tag{4.17}$$

where $\tilde{u}: \mathscr{X} \to \mathscr{X}$ is bounded and continuous. By Lemma 2.1 Eq. (4.17) has a unique weak solution. For $\phi \in BUC_m(\mathscr{X})$ we define $P_t^{\tilde{u}}\phi(X_t^x) = E\phi(x_t^x)$.

PROPOSITION 4.8. *For every $\phi \in BUC_m(\mathscr{X})$ there exists a unique solution g of the integral equation*

$$g(t,x) = P_t\phi(x) - \int_0^t P_{t-s}(\langle \tilde{u}, Dg(s) \rangle)(x)\, ds, \qquad x \in \mathscr{X}, \tag{4.18}$$

such that $g\colon [0,T]\times\mathscr{X}\to R$ *is jointly continuous and* $g(t)\in BUC_m^1(\mathscr{X})$ *for every* $t>0$. *Moreover*, $g(t,x)=P_t^{\tilde u}\phi(x)$ *for* $t\ge 0$, $x\in\mathscr{X}$ *and*

$$\sup_{t\le 1}\left(\sqrt{t}\,\|DP_t^{\tilde u}\phi\|_m\right)+\sup_{t>1}\left(\|DP_t^{\tilde u}\phi\|_m\right)<\infty. \tag{4.19}$$

Proof. We will consider the second term in (4.19) only. Other parts of the proposition may be obtained in a similar way as Theorem 9.24 in [15]. In particular, $\sqrt{t}\,\|DP_t^{\tilde u}\phi\|_m$ is uniformly bounded in $t\le 1$. Taking (4.18) into account we obtain for a fixed $t_0>1$ and $n\ge 1$

$$\begin{aligned} DP_{nt_0}^{\tilde u}\phi(x) &= DP_{nt_0}\phi(x)-\int_0^{nt_0} DP_s\left(\left\langle \tilde u, DP_{nt_0-s}^{\tilde u}\phi\right\rangle\right)(x)\,ds\\ &= DP_{nt_0}\phi(x)-\sum_{k=0}^{n-1}\int_{kt_0}^{(k+1)t_0} DP_s\left(\left\langle \tilde u, DP_{nt_0-s}^{\tilde u}\phi\right\rangle\right)(x)\,ds\\ &= DP_{nt_0}\phi(x)-\sum_{k=0}^{n-1}\int_0^{t_0} DP_{kt_0+s}\left(\left\langle \tilde u, DP_{(n-k)t_0-s}^{\tilde u}\phi\right\rangle\right)(x)\,ds. \end{aligned}$$

Therefore by Proposition 4.1 there exist constants $c_1,c_2,\gamma>0$ such that

$$\begin{aligned} \|DP_{nt_0}^{\tilde u}\phi\|_m &\le \|DP_{nt_0}\phi\|_m+\left(1+\sum_{k=1}^{n-1}\|DP_{kt_0}\|_m\right)\\ &\qquad\times\sup_{x\in\mathscr{X}}\left(\rho_m(x)\int_0^{t_0}\left|P_s\left(\left\langle \tilde u, DP_{(n-k)t_0-s}^{\tilde u}\phi\right\rangle\right)(x)\right|ds\right)\\ &\le c_1\frac{1}{\sqrt{n}}e^{-\gamma n}\|\phi\|_m+c_2\|\phi\|_m\left(1+\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}e^{-\gamma k}\right) \end{aligned}$$

and (4.19) follows. ∎

5. DISCOUNTED COST PROBLEM

In this section we study the discounted cost problem (2.1), (2.4). The formal HJ equation for this has the form

$$\begin{aligned} \alpha v_\alpha(x) &= \frac{1}{2}\operatorname{tr}\left(QD^2v_\alpha(x)\right)+\left\langle Ax+F(x),Dv_\alpha(x)\right\rangle+f(x)\\ &\quad -H(Dv_\alpha(x)), \end{aligned} \tag{5.1}$$

where H is the Hamiltonian of the problem which, under our assumptions, is Lipschitz continuous and convex. It is well known that the requirement on the solution v_α to satisfy (5.1) in the classical sense is too stringent. Therefore, we need to consider the solution to (5.1) in a mild sense. After proving the existence and uniqueness of the mild solution to (5.1) we will identify it with the optimal cost and show that the optimal control can be written in a feedback form.

DEFINITION 5.1. A function $v \in BUC_m^1(\mathscr{X})$ is said to be a solution to Eq. (5.1) if

$$v(x) = \int_0^\infty e^{-\alpha t} P_t(f - H(Dv))(x)\, dt. \tag{5.2}$$

Remark 5.2. It follows from (5.2) that v is a solution to (5.1) if and only if

$$\alpha v = L_m^F v + f - H(Dv). \tag{5.3}$$

Moreover, if (5.3) holds then by part (b) of Corollary 4.6

$$\alpha v(x) = L_{m+1} v(x) + \langle F(x), Dv(x)\rangle + f(x) - H(Dv(x)). \tag{5.4}$$

Equation (5.1) has been considered in [19] under the assumptions that $f \in BUC(\mathscr{X})$ and F is bounded. The lemma below extends the results of [19] to the case of unbounded F.

LEMMA 5.3. *Assume that $f \in BUC(\mathscr{X})$. Then the following holds.*

(a) *For every $\alpha > 0$ there exists a unique solution $v_\alpha \in BUC^1(\mathscr{X})$ of Eq.* (5.1).

(b) *The solution v_α may be identified with the value function for the control problem* (2.1), (2.3). *Moreover, $\hat{u}_\alpha = DH(Dv_\alpha)$ is the optimizing feedback control.*

(c) *For every $\alpha > 0$ and $m \geq 0$ there exist $a, b > 0$ such that*

$$\|Dv_\alpha\|_m \leq a\|f\|_m + b.$$

(d) *For every $m \geq 0$ and $\theta \in (0,1)$ sufficiently small there exists $a_1 = a_1(m,\theta) > 0$ such that*

$$\|Dv_\alpha\|_{\theta, m+1} \leq a_1(\|f\|_m + \|Dv_\alpha\|_m). \tag{5.5}$$

Proof. Taking (A2) into account parts (a) and (b) follow by an easy modification of the proof of Theorem 3.11 in [19].

(c) By (P_t^α) we denote the transition semigroup corresponding to the closed loop equation

$$\begin{cases} dX_t^\alpha = \left(AX_t^\alpha + F(X_t^\alpha) - \hat{u}_\alpha(X_t^\alpha)\right) dt + \sqrt{Q}\, dW_t, \\ X_0 = x, \end{cases}$$

where $\hat{u}_\alpha$ is the optimal control introduced in (b). Since v_α^f is the optimal cost for the discounted cost control problem (2.1), (2.3), we have

$$v_\alpha(x) = \int_0^\infty e^{-\alpha t} P_t^\alpha (f + h(\hat{u}_\alpha))(x)\, dt$$

and therefore by Proposition 4.8 there exists $c > 0$ such that

$$\begin{aligned} \|Dv_\alpha\|_m &\leq \| f + h(\hat{u}_\alpha) \|_m \\ &\qquad \times \left(\sup_{t \leq 1} \left(\sqrt{t}\, \|DP_t^\alpha\|_m \right) \int_0^1 \frac{1}{\sqrt{t}}\, dt + \sup_{t > 1} \|DP_t^\alpha\|_m \int_1^\infty e^{-\alpha t}\, dt \right) \\ &\leq c \left(\|f\|_m + \sup_{\|z\| \leq r} |h(z)| \right) \end{aligned}$$

which yields (c).

(d) Invoking part (b) of Lemma 3.2 we find that

$$\begin{aligned} \|Dv_\alpha\|_{\theta, m+1} &\leq C(\theta) \int_0^\infty M_\theta(t) \| \langle F, Dv_\alpha \rangle + f - H(Dv_\alpha) \|_{m+1}\, dt \\ &\leq \left(c_1 \|F\|_1\, \|Dv_\alpha\|_m + c_2 \|f\|_m + c_3 \right. \\ &\qquad\qquad \left. + c_4 \|Dv_\alpha\|_m \right) C(\theta) \int_0^\infty M_\theta(t)\, dt \end{aligned}$$

for some constants c_0, c_3, c_4, c_5, and c_6 independent of $\alpha > 0$ and f which yields (5.5). ∎

LEMMA 5.4. *For every $m \geq 0$ and $\alpha > 0$ and $f \in BUC_m(\mathscr{X})$ there exists a solution v_α to Eq.* (5.1). *Moreover*, (5.5) *holds for v_α.*

Proof. In the proof below we write for the sake of simplicity v instead of v_α. Let $\alpha > 0$ and $m \geq 0$ be fixed and let

$$f_n(x) = \begin{cases} f(x) & \text{if } \|x\| \leq n, \\ f\left(\dfrac{nx}{\|x\|} \right) & \text{if } \|x\| > n. \end{cases}$$

By Lemma 5.3 there exists a unique solution $v_n \in BUC(\mathscr{X})$ to (5.1) corresponding to f_n and

$$|v_n(x)| \le J_\alpha^F(|f_n - H(Dv_n)|)(x), \qquad x \in \mathscr{X}. \tag{5.6}$$

Since $\sup_{n\ge 1}\|f_n - H(Dv_n)\|_\mu < \infty$ by (c) of Lemma 5.3 and (5.6) it follows that

$$\sup_{n\ge 1}\left(\|v_n\|_\mu + \|Dv_n\|_\mu\right) < \infty.$$

Hence, putting $\psi_n = f_n + \langle F, Dv_n\rangle - H(Dv_n)$ we obtain $\sup_{n\ge 1}\|\psi_n\|_\mu < \infty$. Since $v_n = J_\alpha \psi_n$ Lemma 3.1 implies that the families $\{v_n\colon n \ge 1\}$ and $\{Dv_n\colon n \ge 1\}$ are relatively compact in $L^2(\mathscr{X}, \mu)$ and $L^2(\mathscr{X}, \mu; \mathscr{X})$, respectively. Therefore, for a certain subsequence, still denoted by v_n,

$$\lim_{n\to\infty} v_n = v \quad \text{and} \quad \lim_{n\to\infty} Dv_n = w$$

for certain $v \in L^2(\mathscr{X}, \mu)$ and $w \in L^2(\mathscr{X}, \mu; \mathscr{X})$. Finally, since D is closable in $L^2(\mathscr{X}, \mu)$ we obtain $v \in W^{1,2}(\mathscr{X}, \mu)$ and $w = Dv$. Since

$$v_n(x) = \int_{\mathscr{X}}\left(\int_0^\infty e^{-\alpha t}G(t,x,y)\,dt\right)\left(f_n(y) - H(Dv_n(y))\right)\mu(dy)$$

we find that v_n converges to v for every $x \in \mathscr{X}$ and by Lemma 5.3 and the Lipschitz property of H

$$\begin{aligned}
&\rho_m(x)|v_n(x)| \\
&\quad \le \rho_m(x)\int_{\mathscr{X}}\left(\int_0^\infty e^{-\alpha t}G(t,x,y)\,dt\right)\rho_m^{-1}(y)\left(a_1\|f\|_m + b_1\right)\mu(dy) \\
&\quad = \left(a_1\|f\|_m + b_1\right)\rho_m(x)J_\alpha\rho_m^{-1}(x)
\end{aligned} \tag{5.6}$$

for certain $a_1, b_1 > 0$. Therefore, passing with n to infinity we find that for a certain $c > 0$

$$\|v\|_m \le c(1 + \|f\|_m).$$

Since

$$\begin{aligned}
|\rho_m(x)v_n(x) - \rho_m(y)v_n(y)| \le (\rho_m(x)& \\
-\rho_m(y))v_n(x) + \rho_m(y)&|v_n(x) - v_n(y)|
\end{aligned}$$

Lemma 5.3 and (5.6) yield for a certain $c_2 > 0$

$$\begin{aligned}
&|\rho_m(x)v_n(x) - \rho_m(y)v_n(y)| \\
&\quad \le (\rho_m(x) - \rho_m(y))\rho_m^{-1}(x)c_2 \\
&\qquad + \|x - y\|\,\rho_m(y)\sup_{0\le\zeta\le 1}\|Dv_n(x + \zeta(y - x))\|.
\end{aligned}$$

Hence, invoking Lemma 5.3 again and proceeding in the same way as in the proof of Proposition 4.1 we find that the family $\{\rho_m v_n\colon n \geq 1\}$ is equi-uniformly continuous and thereby v has uniformly continuous modification. Similarly, (d) of Lemma 5.3 implies that Dv_n are Hölder-continuous uniformly in $n \geq 1$. Hence, $\rho_m Dv$ has a uniformly continuous modification and $Dv \in BUC_m(\mathscr{X})$ again by Lemma 5.3. ∎

THEOREM 5.5. *For every $f \in BUC_m(\mathscr{X})$ and $\alpha > 0$ the value function $V_\alpha(x) = \inf_{u \in \mathscr{U}} V_\alpha(x,u)$ is a unique solution of Eq.* (5.1). *Moreover, the optimal state $\hat{X}_t$ solves Eq.* (2.1) *with the optimal control $\hat{u}(\hat{X}_t) = DH(Dv_\alpha(\hat{X}_t))$.*

Proof. The proof is a simple modification of the proof given in [19] for the case of bounded function F and f. The main ingredient is the identity

$$v_\alpha(x) + E_{x,u}\left(\int_0^\infty e^{-\alpha t}\big(H(Dv_\alpha(X_t)) - \langle u_t, Dv_\alpha(X_t)\rangle + h(u_t)\big)\,dt\right)$$
$$= V_\alpha(x,u)$$

which holds for every $u \in \mathscr{U}$ and $x \in \mathscr{X}$. This identity has been proved in [19] for the case of bounded function f and F. The case of F Lipschitz and $f \in BUC_m(\mathscr{X})$ follows by approximation due to Corollary 4.6 and Lemma 3.4. ∎

6. HJ EQUATION FOR THE ERGODIC CONTROL PROBLEM

In this section we assume, additional to (A1)–(A4), the following condition

$$\text{(A5)} \qquad r < \frac{\sqrt{\omega_1}}{Ck(\omega_1)\sqrt{\pi}} \quad \text{for some } \omega_1 \in (0, \omega - \beta),$$

where $k(\omega_1)$ and C are defined in Proposition 4.1.

For a Markov control $\tilde{u} \in \tilde{\mathscr{U}}$ denote by $(P_t^{\tilde{u}})$ the transition semigroup corresponding to Eq. (2.2) and let

$$\tilde{\mathscr{U}}_c = \{\tilde{u} \in \tilde{\mathscr{U}}\colon \tilde{u} \text{ is continuous}\}.$$

PROPOSITION 6.1. *There exists a function $\lambda \in L^1(0,\infty)$ such that*

$$\|DP_t^{\tilde{u}}\phi\|_m \leq \lambda(t)\|\phi\|_m, \qquad t > 0,$$

for each $\tilde{u} \in \tilde{\mathscr{U}}_c$ and $\phi \in BUC_m(\mathscr{X})$.

Proof. Since $\tilde{u}$ is bounded, Proposition 4.8 yields $P_t^{\tilde{u}}\phi \in BUC_m^1(\mathscr{X})$ for all $t > 0$ and

$$P_t^{\tilde{u}}\phi(x) = P_t\phi(x) + \int_0^t P_{t-s}(\langle -\tilde{u}(\cdot), DP^{\tilde{u}}{}_s\phi(\cdot)\rangle)(x)\,ds. \quad (6.1)$$

Since the function $t \to \|DP_t^{\tilde{u}}\phi\|_m$ is measurable (6.1) and (4.5) yield for $0 < \omega_1 < \omega - \beta$

$$\|DP_t^{\tilde{u}}\phi\|_m \leq \frac{Ck(\omega_1)}{\sqrt{t}} e^{-\omega_1 t}\|\phi\|_m + \int_0^t \frac{Crk(\omega_1)}{\sqrt{t-s}} e^{-\omega_1(t-s)}\|DP_s^{\tilde{u}}\phi\|_m\,ds.$$

Now we can use Lemma 2.4 with $a = \omega_1$, $c_2 = rCk(\omega_1)$, $c_1 = Ck(\omega_1)\|\phi\|_m$ and $\beta = \frac{1}{2}$ to obtain

$$\|DP_t^{\tilde{u}}\phi\|_m \leq Ck(\omega_1)\gamma(t)\|\phi\|_m, \quad (6.2)$$

where $\gamma \in L^1(0,\infty)$ is given by Lemma 2.4. ∎

The formal HJ equation for the problem (2.1), (2.4) has the form

$$\frac{1}{2}\operatorname{tr}(QD^2 v(x)) + \langle Ax + F(x), Dv(x)\rangle + f(x) - H(Dv(x)) - \lambda = 0. \quad (6.3)$$

DEFINITION 6.2. A pair $(v, \lambda) \in BUC_m^1(\mathscr{X}) \times R$ is called a solution to Eq. (6.3) in the space $BUC_m^1(\mathscr{X}) \times R$ if $v \in \operatorname{dom}(L_m)$ and for every $x \in \mathscr{X}$

$$L_m v(x) + \langle F(x), Dv(x)\rangle + f(x) - H(Dv(x)) - \lambda = 0. \quad (6.4)$$

LEMMA 6.3. *The following conditions are equivalent*:

(a) $v \in BUC_m^1(\mathscr{X})$,

$$\int_{\mathscr{X}} (\langle F(x), Dv(x)\rangle + f(x) - H(Dv(x)) - \lambda)\mu(dx) = 0 \quad (6.5)$$

and

$$v(x) = \int_0^\infty R_t(\langle F, Dv\rangle + f - H(Dv) - \lambda)(x)\,dt \quad (6.6)$$

with the integral converging absolutely for every $x \in \mathscr{X}$.

(b) *The pair* (v, λ) *is a solution to* (6.3) *in* $BUC_m^1(\mathscr{X}) \times R$.

Proof. It easy to check the (a) implies (b). Assume that (b) holds and define

$$\psi(x) = \langle F(x), Dv(x)\rangle + f(x) - H(Dv(x)) - \lambda.$$

Let L_μ denote the generator of the semigroup (R_t) when considered in $L^2(\mathscr{X}, \mu)$. The operator L_μ is an extension of L_m; hence $v \in \operatorname{dom}(L_\mu)$. Since μ is an invariant measure for the semigroup (R_t) we have $\langle L_\mu v, 1\rangle = 0$ and by the definition of v we find that $\langle \psi, 1\rangle_\mu = 0$ which gives (6.5). Then invoking Lemma 3.1(b) we find that

$$\int_0^\infty |R_t \psi(x)|\, dt < \infty \qquad \mu\text{-a.s.}$$

and since $R_t\psi(x)$ is continuous in (t, x) the absolute convergence holds for all $x \in \mathscr{X}$. Putting

$$u_\alpha(x) = \int_0^\infty e^{-\alpha t} R_t \psi(x)\, dt$$

we find that u_α is $\mathscr{K}$-convergent to u_0 for $\alpha \to 0$ and

$$(\alpha - L_m) u_\alpha = \psi.$$

Since L_m is $\mathscr{K}$-closed in $BUC_m(\mathscr{X})$ (6.6) follows. ∎

To prove existence of a solution to (6.3) we will need first uniform estimates on solution to the HJ equation (5.1).

LEMMA 6.4. *Let v_α be the unique solution to* (5.1) *in $BUC_m(\mathscr{X})$. Then*

$$\sup_{\alpha > 0} \|Dv_\alpha\|_m < \infty.$$

Proof. Since $v_\alpha(x)$ is the optimal cost for the discounted control problem (2.1), (2.3), we have

$$V_\alpha(x) = \int_0^\infty e^{-\alpha t} P_t^\alpha (f + h(\hat{u}_\alpha))(x)\, dt, \tag{6.7}$$

by Theorem 5.5, where P_t^α is the transition semigroup corresponding to the closed loop equation

$$\begin{cases} dX_t^\alpha = (AX_t^\alpha + F(X_t^\alpha) - \hat{u}_\alpha(X_t^\alpha))\, dt + \sqrt{Q}\, dW_t, \\ X_0^\alpha = x, \end{cases} \tag{6.8}$$

and $\hat{u}_\alpha = DH(Dv_\alpha)$ is the optimal control. By Theorem 5.5 $\hat{u}_\alpha \in \tilde{\mathscr{U}}_c$; therefore Proposition 6.1 and (6.7) yield

$$\|Dv_\alpha\|_m \le \sup_{x \in \mathscr{X}} \left| \rho_m(x) \int_0^\infty e^{-\alpha t} DP_t^\alpha (f + h(\hat{u}_\alpha))(x)\, dt \right|$$

$$\leq \sup_{x \in \mathscr{X}} \left(\rho_m(x) \int_0^\infty \| DP_t^\alpha (f + h(\hat{u}_\alpha))(x) \| \, dt \right)$$

$$\leq \| f + h(\hat{u}_\alpha) \|_m \int_0^\infty \lambda(t) \, dt$$

$$\leq \left(\|f\|_m + 2 \sup_{\|z\| \leq r} |h(z|\right) \int_0^\infty \lambda(t) \, dt.$$

∎

Remark 6.5. Assume that $\tilde{F} = F - \beta I \in BUC(\mathscr{X})$ with $\beta < \omega$. Then Remark 5.3 allows us to find a uniform in $\alpha > 0$ estimate on $\|Dv_\alpha\|_0$ in a way different from Lemma 6.4. For $\tilde{R}_t = e^{\beta t} R_t$ and $\phi \in BUC(\mathscr{X})$ the norms $\|D\tilde{R}_t \phi\|_0$ can be estimated as follows:

$$\|D\tilde{R}_t \phi\|_0 \leq \frac{k(\omega_1)}{\sqrt{\|Q\|}} \frac{1}{\sqrt{t}} e^{-\omega_1 t} \tag{6.9}$$

for $0 < \omega_1 < \omega - \beta$, where $k(\omega_1)$ is defined in Proposition 4.2. Since (5.4) yields

$$\tilde{L} v_\alpha + \langle \tilde{F}, Dv_\alpha \rangle + f - H(Dv_\alpha) = \alpha v_\alpha,$$

where $\tilde{L}$ is the generator of the semigroup $\tilde{R}_t$ in $BUC(\mathscr{X})$, we have

$$\tilde{L} v_\alpha + \langle \tilde{F}, Dv_\alpha \rangle - \langle \hat{u}_\alpha, Dv_\alpha \rangle + f + h(\hat{u}_\alpha) = \alpha v_\alpha \tag{6.10}$$

and therefore

$$v_\alpha(x) = \int_0^\infty e^{-\alpha t} \tilde{R}_t \Big(\langle \tilde{F}, Dv_\alpha \rangle - \langle \hat{u}_\alpha, Dv_\alpha \rangle + f + h(\hat{u}_\alpha) \Big)(x) \, dt. \tag{6.11}$$

Differentiating in (6.11) and taking into account (6.9) we get

$$\|Dv_\alpha\|_0 \leq k_1 + \left(\frac{k(\omega_1)}{\sqrt{\|Q\|}} \big(\|\tilde{F}\|_0 + r \big) \int_0^\infty t^{-1/2} e^{-\omega_1 t} \, dt \right) \|Dv_\alpha\|_0, \tag{6.12}$$

where k_1 is a constant. Thus setting

$$B(\omega_1) = \int_0^\infty t^{-1/2} e^{-\omega_1 t}\, dt$$

we find that $\|Dv_\alpha\|_0$ are bounded uniformly in $\alpha > 0$ provided

$$r + \|\tilde{F}\|_0 < \frac{\sqrt{\|Q\|}}{k(\omega_1) B(\omega_1)} \tag{6.13}$$

for some $0 < \omega_1 < \omega - \beta$. So in this case $\|\tilde{F}\|_0$ has to be small enough but it does not have to be dissipative. ∎

Set

$$c(\alpha) = \frac{1}{\alpha} \int_{\mathscr{X}} (f + \langle F, Dv_\alpha \rangle - H(Dv_\alpha))(x)\, \mu(dx) \tag{6.14}$$

and $\bar{v}_\alpha = v_\alpha - c(\alpha)$. We shall prove the existence of a solution to Eq. (6.3) by sapsage to the limit in the sequence $(\bar{v}_\alpha, \alpha c(\alpha))$.

THEOREM 6.6. *Let $f \in B\mathscr{U}C_m(\mathscr{X})$. Then there exists a solution (v, λ) to Eq.* (6.3) *in the space $BUC^1_{m+1}(\mathscr{X}) \times R$.*

Proof. *Step 1.* We show first that the equation

$$v = \int_0^\infty R_t (f + \langle F, Dv \rangle - H(Dv) - \lambda)\, dt$$

has a solution in $L^2(\mathscr{X}, \mu)$. Let

$$\psi_\alpha = f + \langle F, Dv_\alpha \rangle - H(Dv_\alpha)$$

and

$$\bar{\psi}_\alpha = \psi_\alpha - \langle \psi_\alpha, 1 \rangle_\mu.$$

Then by Lemma 6.4

$$\sup_\alpha \|\bar{\psi}_\alpha\|_\mu < \infty \tag{6.15}$$

and $\bar{v}_\alpha = J_\alpha \bar{\psi}_\alpha$. Hence

$$\|\bar{v}_\alpha\|_\mu \le \int_0^\infty \|R_t \bar{\psi}_\alpha\|_\mu\, dt \le c \|\bar{\psi}_\alpha\|_\mu. \tag{6.16}$$

Similarly, in view of Lemma 3.1 we find that

$$\|D\bar{v}_\alpha\|_\mu \le \int_0^\infty \|DR_t\|_\mu \, dt \, \|\bar{\psi}_\alpha\|_\mu . \tag{6.17}$$

We will show that the family $\{v_\alpha: \alpha > 0\}$ is relatively compact in $L^2(\mathscr{X}, \mu)$. Indeed,

$$\bar{v}_\alpha = \int_0^\infty (e^{-\alpha t} - 1) R_t \bar{\psi}_\alpha \, dt + \int_0^\infty R_t \bar{\psi}_\alpha \, dt \tag{6.18}$$

and by (6.15)

$$\lim_{\alpha \to 0} \int_0^\infty (e^{-\alpha t} - 1) R_t \bar{\psi}_\alpha \, dt = 0$$

uniformly in $\bar{\psi}_\alpha$. On the other hand Lemma 3.1 and (6.15) yield relative compactness of the family

$$\left\{ \int_0^\infty R_t \bar{\psi}_\alpha \, dt \colon \alpha > 0 \right\}$$

and the claim follows. Similarly, combining Lemma 3.1, (6.15), and (6.17) we find that the set $\{D\bar{v}_\alpha: \alpha > 0\}$ is relatively compact in $L^2(\mathscr{X}, \mu; \mathscr{X})$. Since the operator D with the domain $W^{1,2}(\mathscr{X}, \mu)$ is closed there exists a subsequence $\alpha_n \to 0$ such that

$$\bar{v}_{\alpha_n} \to v \quad \text{in } L^2(\mathscr{X}, \mu), \qquad D\bar{v}_{\alpha_n} \to Dv \quad \text{in } L^2(\mathscr{X}, \mu; \mathscr{X})$$

for a certain $v \in W^{1,2}(\mathscr{X}, \mu)$ (note that $D\bar{v}_\alpha$ and Dv_α coincide because $\bar{v}_\alpha$ and v_α differ just by a constant). Also, by Lemma 6.4 it is obvious that $|\alpha c(\alpha)|$ is bounded for $\alpha > 0$; therefore $\alpha_n c(\alpha_n) \to 0$ for a subsequence $\alpha_n \to 0$ and a certain $\lambda \in R$. It follows that in $L^2(\mathscr{X}, \mu)$

$$v = \int_0^\infty R_t(f + \langle F, Dv \rangle - H(Dv) - \lambda) \, dt .$$

We shall check that the pair (v, λ) satisfies Eq. (6.3) in $L^2(\mathscr{X}, \mu)$. Since $v_\alpha \in \operatorname{dom}(L_\mu)$, where L_μ denotes the generator of the semigroup (R_t) in $L^2(\mathscr{X}, \mu)$, and v_α satisfies (5.4) we find that

$$L_\mu \bar{v}_\alpha + \langle F, D\bar{v}_\alpha \rangle - H(D\bar{v}_\alpha) + f - \alpha \bar{v}_\alpha - \alpha c(\alpha) = 0 \tag{6.19}$$

for $\alpha > 0$. By the first part of the proof

$$H\left(D\bar{v}_{\alpha_n}\right) \to H(Dv), \qquad \alpha_n \bar{v}_{\alpha_n} \to 0 \tag{6.20}$$

in $L^2(\mathscr{X}, \mu)$. Choosing possibly a subsequence (again denoted by α_n) we find that $D\bar{v}_{\alpha_n} \to Dv$ μ-a.e., so by the Dominated Convergence Theorem we obtain

$$\langle F, D\bar{v}_{\alpha_n} \rangle \to \langle F, Dv \rangle \quad \text{in } L^2(\mathscr{X}, \mu). \tag{6.21}$$

By (6.20) and (6.21) Eq. (6.19) implies that $L_\mu \bar{v}_{\alpha_n} \to \psi$ in $L^2(\mathscr{X}, \mu)$ for a certain $\psi \in L^2(\mathscr{X}, \mu)$ and since L_μ is closed in $L^2(\mathscr{X}, \mu)$ we conclude that $v \in \mathrm{dom}(L_\mu)$ and

$$L_\mu v = -\langle F, Dv \rangle + H(Dv) - f + \lambda \tag{6.22}$$

so that (v, λ) is a solution to equation (6.4) in $L^2(\mathscr{X}, \mu)$.

Step 2: Existence in $BUC_{m+1}(\mathscr{X}) \times R$.

Let (α_n) be such a sequence that

$$\bar{v}_{\alpha_n} \to v \quad \text{and} \quad D\bar{v}_{\alpha_n} \to Dv \qquad \mu\text{-a.s.} \tag{6.23}$$

Invoking the proof of Lemma 5.4 and Lemma 6.4 we find that $\{\rho_m v_\alpha : \alpha > 0\}$ is a family of functions which are uniformly continuous uniformly in $\alpha > 0$ and thereby $\rho_m v$ has a uniformly continuous modification and $\|v\|_{m+1} < \infty$. Similarly, (6.22) yields Hölder continuity of Dv uniformly on bounded sets. Hence $\|Dv\|_{m+1} < \infty$ and $\rho_{m+1} Dv$ is uniformly continuous. It follows that $v \in BUC^1_{m+1}(\mathscr{X})$ and by Lemma 6.3 (v, λ) is a solution to (6.3) in $BUC_{m+1}(\mathscr{X}) \times R$. ∎

Let (v, λ) be any solution to (6.3). The next theorem provides the characterization of λ. Note that in the result below we do not need condition (A5).

THEOREM 6.7. *Let (v, λ) be any solution to Eq.* (6.3). *Then*

$$\lambda = \inf_{u \in \mathscr{U}} \tilde{J}(x, u), \qquad x \in \mathscr{X},$$

that is, λ is the optimal cost, hence unique. The feedback control $\hat{u}(x) = DH(Dv(x))$ is an optimal control for the problem (2.1), (2.3).

Proof. Take a sequence $(v_n) \subset \mathrm{dom}(L_0)$ satisfying the conditions of Lemma 3.5 with ϕ and ϕ_n replaced by v and v_n respectively. Set

$$f_n(x) = \lambda + H\big(Dv_n(x)\big) - \big\langle F(x), Dv_n(x) \big\rangle - L_0 v_n(x).$$

Lemma 3.5 implies that for a certain subsequence still denoted by n f_n is $\mathscr{K}$-convergent to f and

$$\big| f_n(x) \big| \le k\big(1 + \|x\|^{m+1}\big), \qquad x \in \mathscr{X}.$$

Applying the Ito formula to the solution of Eq. (2.1) with $u \in \mathscr{U}$ and the function $-\lambda t + v_n(x)$ we obtain for $x \in \mathscr{X}$ and $t > 0$

$$-\lambda t + E_{x,u} v_n(X_t) = v_n(x) + E_{x,u} \int_0^t \big(-\lambda + L_0 v_n(X_s) + \langle F(X_s), Dv_n(X_s) \rangle - \langle u_s, Dv_n(X_s) \rangle \big)\, ds.$$

Thus

$$\lambda t = E_{x,u} v_n(X_t) - v_n(x) + E_{x,u} \int_0^t \big(f_n(X_s) - H(Dv_n(X_s)) + \langle u_s, Dv_n(X_s) \rangle \big)\, ds. \tag{6.24}$$

Passing with n to infinity we find that

$$\lambda t = E_{x,u} v(X_t) - v(x) + E_{x,u} \int_0^t \big(f(X_s) - H(Dv(X_s)) + \langle u_s, Dv(X_s) \rangle \big)\, ds.$$

By the definition of H

$$f(X_s) - H(Dv(X_s)) + \langle u_s, Dv(X_s) \rangle \le f(X_s) + h(u_s)$$

with the equality attained for $u_s = DH(Dv(X_s))$. Hence, invoking Lemma 2.2 we obtain for $t > 0$ and $x \in \mathscr{X}$

$$\lambda \le \frac{1}{t} E_{x,u}\, v(X_t) - \frac{k}{t}(1 + \|x\|^\gamma) + \frac{1}{t} E_{x,u} \int_0^t \big(f(X_s) + h(u_s) \big)\, ds$$

and

$$\lambda = \frac{1}{t} E_{x,\hat{u}}\, v(X_t) - \frac{1}{t} v(x) + \frac{1}{t} E_{x,\hat{u}} \int_0^t \big(f(X_s) + h(\hat{u}(X_s)) \big)\, ds.$$

Passing with t to infinity we find that

$$\lambda \le \liminf_{t \to \infty} \frac{1}{t} E_{x,u} \int_0^t \big(f(X_s) + h(u_s) \big)\, ds$$

and

$$\lambda = \liminf_{t \to \infty} \frac{1}{t} E_{x,\hat{u}} \int_0^t \big(f(X_s) + h(\hat{u}(X_s)) \big)\, ds. \tag{6.25}$$

The limit in (6.25) exists and is equal to

$$\lambda = \int_{\mathscr{X}} (f(y) + h(\hat{u}(y)))\,\hat{\mu}(dy)$$

by the Strong Law of Large Numbers (cf. [23]) for all $x \in \mathscr{X}$, where $\hat{\mu}$ is the invariant measure of the closed loop equation (2.1) with $u = \hat{u}$. ▮

By Theorem 6.7, if (v, λ) is a solution to (6.3) then λ is the optimal cost for the ergodic control problem (2.1), (2.4); hence λ is uniquely determined. In the following theorem we show that v is also uniquely determined up to an additive constant (clearly, $(v + c, \lambda)$ is also a solution of the ergodic control problem for any $c \in R$).

THEOREM 6.8. *For any $m \geq m_0 + 1$ Eq.* (6.3) *has exactly one solution $(v, \lambda) \in BUC_m^1(\mathscr{X}) \times R$ such that $v(0) = 0$. Moreover,*

$$v(x) = \lim_{r \to 0+} \inf_{u \in \tilde{\mathscr{U}}} \left(\overline{\lim_{t \to \infty}} E_{x,u} \int_0^{\tau_r \wedge t} \big(f(X_s) + h(u(X_s)) - \lambda\big)\, ds \right), \quad (6.26)$$

where τ_r stands for the first hitting time of the ball B_r.

Proof. In the proof p is a large enough generic constant which can may be different in different formulas. Let (w, λ) be any solution to (6.3) such that $w \in BUC_m(\mathscr{X})$. Similarly as in the proof of Theorem 6.7 we find a sequence $w_n \in \mathscr{D}_0$ such that

$$w_n \to w, \qquad Dw_n \to Dw, \qquad L_k w_n \to L_k w$$

uniformly on compact sets in $\mathscr{X}$ and

$$\|Dw_n\|_m \leq c_1, \qquad |L_m w_n(x)| \leq c_2(1 + \|x\|^p) \quad (6.27)$$

for some c_1, c_2, p large enough, independent of n. Clearly (w_n, λ) solves Eq. (6.3) with f replaced with

$$f_n = \lambda + H(Dw_n) - \langle F, Dw_n \rangle - L_k w_n.$$

Putting $\bar{u}_s = DH(Dw(X_s))$ we obtain by the Ito formula

$$E_{x,\bar{u}} w_n(X_{\tau_r \wedge t}) = w_n(x) + E_{x,\bar{u}} \int_0^{\tau_r \wedge t} \big(H(Dw_n(X_s)) - f_n(X_s) - \langle \bar{u}_s, Dw_n(X_s)\rangle + \lambda\big)\, ds \quad (6.33)$$

for $x \in \mathscr{X}$, $t > 0$ and $r > 0$. Hence, passing with n to infinity and taking into account the definition of H we find that

$$E_{x,\bar{u}} w(X_{\tau_r \wedge t}) = w(x) + E_{x,\bar{u}} \int_0^{\tau_r \wedge t} (-f(X_s) - h(\bar{u}_s) + \lambda)\, ds.$$

Furthermore, note that $|w(x)| \leq c_3 \|x\|$ for $\|x\| \leq 1$ and

$$|w(x)| \leq c_4(1 + \|x\|^p), \qquad x \in \mathscr{X}$$

for some $c_3, c_4 > 0$ by (6.27), and the solution X of (2.2) is recurrent for each Markov control (cf. [24]). Thus

$$E_{x,\bar{u}} w(X_{\tau_r \wedge t}) = E_{x,\bar{u}}\big(I_{(\tau_r \leq t)} w(X_{\tau_r}) + I_{(\tau_r > t)} w(X_t)\big) \to E_{x,\bar{u}} w(X_{\tau_r}) \tag{6.28}$$

as $t \to \infty$ by the Dominated Convergence Theorem and Lemma 2.2 (note that $|w(X_{\tau_r})| \leq c_3 r$ a.s. for $r < 1$). Consequently

$$\begin{aligned} w(x) &\geq \varlimsup_{t \to \infty} E_{x,\bar{u}} \int_0^{\tau_r \wedge t} (f(X_s) + h(\bar{u}(X_s)) - \lambda)\, ds - c_1 r \\ &\geq \inf_{u \in \tilde{\mathscr{U}}} \varlimsup_{t \to \infty} E_{x,u} \int_0^{\tau_r \wedge t} (f(X_s) + h(u(X_s)) - \lambda)\, ds - c_1 r. \end{aligned}$$

Hence

$$w(x) \geq \varlimsup_{r \to 0+} \inf_{u \in \tilde{\mathscr{U}}} \varlimsup_{t \to \infty} E_{x,u} \int_0^{\tau_r \wedge t} (f(X_s) + h(u(X_s)) - \lambda)\, ds. \tag{6.29}$$

Now, let (v, λ) be the solution to Eq. (6.3) found in Theorems 6.6 and 6.7 (we preserve the notation introduced in those theorems). For each $\alpha > 0$ there exists, by Lemma 3.4, a sequence $(v_\alpha^n) \subset \mathscr{D}_0$ such that

$$v_\alpha^n \to v_\alpha, \qquad Dv_\alpha^n \to Dv, \qquad L_m v_\alpha^n \to L_m v_\alpha$$

uniformly on compact sets in $\mathscr{X}$ and for a certain $p > 0$

$$\|Dv_\alpha^n\|_m \leq k_1, \qquad |v_\alpha^n(x)| + |L_m v_\alpha^n(x)| \leq k_2(1 + \|x\|^p)$$

for some $k_1, k_2 > 0$. Define

$$f_\alpha^n = \alpha v_\alpha^n + H(Dv_\alpha^n) - \langle F, Dv_\alpha^n \rangle - L_m v_\alpha^n.$$

Clearly, $f_\alpha^n \to f$ as $n \to \infty$ for each $\alpha > 0$, the convergence being uniform on compat sets in $\mathscr{X}$ and

$$|f_\alpha^n(x)| \leq k_3(1 + \|x\|^p)$$

for some $k_3,\ p > 0$. By the Ito formula we have for each $u \in \tilde{\mathscr{U}}$

$$
\begin{aligned}
E_{x,u} e^{-\alpha(\tau_r \wedge t)} v_\alpha^n\big(X_{\tau_r \wedge t}\big) &- c(\alpha) - v(0) \\
&\le E_{x,u} e^{-\alpha(\tau_r \wedge t)} v_\alpha\big(X_{\tau_r \wedge t}\big) \\
&\quad + E_{x,u} \int_0^{\tau_r \wedge t} e^{-\alpha s}\big(f(X_s) + h(u(X_s))\big)\, ds - c(\alpha) - v(0).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
v_\alpha(x) &- c(\alpha) - v(0) \\
&\le E_{x,u} \int_0^{\tau_r \wedge t} e^{-\alpha s}\big(f(X_s) + h(u(X_s)) - \alpha v_\alpha\big(X_{\tau_r \wedge t}\big)\big)\, ds \\
&\quad + E_{x,u} v_\alpha\big(X_{\tau_r \wedge t}\big) - c(\alpha) - v(0).
\end{aligned}
$$

Since

$$
\alpha v_\alpha = L_m v_\alpha + \langle F, D v_\alpha \rangle - H(D v_\alpha) + f,
$$

$$
D v_\alpha = D \bar{v}_\alpha \to D v \quad \text{and} \quad L_m \bar{v}_\alpha = L_m v_\alpha \to L_m v
$$

uniformly on compacts and all the function above are polynomially bounded we have for every $x \in \mathscr{X}$, $t > 0$ and $r > 0$

$$
\lim_{\alpha \to 0} E_{x,u} \alpha v_\alpha\big(X_{\tau_r \wedge t}\big) = \lambda
$$

and from (6.23) and the local equicontinuity of (v_α) it follows that

$$
E_{x,u} v_\alpha\big(X_{\tau_r \wedge t}\big) - c(\alpha) = E_{x,u} \bar{v}_\alpha\big(X_{\tau_r \wedge t}\big) \to E_{x,u} v\big(X_{\tau_r \wedge t}\big)
$$

as $\alpha \to 0$. Therefore

$$
\begin{aligned}
v(x) - v(0) &\le E_{x,u} \int_0^{\tau_r \wedge t} \big(f(X_s) + h(u(X_s)) - \lambda\big)\, ds \\
&\quad + E_{x,u} v\big(X_{\tau_r \wedge t}\big) - v(0)
\end{aligned}
$$

and taking $t \to \infty$ similarly as in (6.28) we obtain for $0 < r < 1$ and a constant $c_5 > 0$

$$
\begin{aligned}
v(x) - v(0) &\le \limsup_{t \to \infty} E_{x,u} \int_0^{\tau_r \wedge t} \big(f(X_s) + h(u(X_s)) - \lambda\big)\, ds \\
&\quad + E_{x,u} v\big(X_{\tau_r}\big) - v(0) \\
&\le \limsup_{t \to \infty} E_{x,u} \int_0^{\tau_r \wedge t} \big(f(X_s) + h(u(X_s)) - \lambda\big)\, ds + c_5 r.
\end{aligned}
$$

Hence

$$v(x) - v(0)$$

$$\leq \liminf_{r\to 0} \inf_{u\in\tilde{\mathscr{U}}} \limsup_{t\to\infty} E_{x,u} \int_0^{\tau_r\wedge t} \big(f(X_s) + h(u(X_s)) - \lambda\big)\, ds$$

which together with (6.29) (where $w = v - v(0)$) yields

$$v(x) - v(0)$$

$$= \lim_{r\to 0+} \inf_{u\in\tilde{\mathscr{U}}} \limsup_{t\to\infty} E_{x,u} \int_0^{\tau_r\wedge t} \big(f(X_s) + h(u(X_s)) - \lambda\big)\, ds$$

and

$$w \geq v - v(0) \tag{6.30}$$

for each w, satisfying the assumptions of the theorem. It remains to prove a converse inequality. Setting $\tilde{v} = v - v(0)$, $\tilde{u} = DH(D\tilde{v})$ and $U = w - \tilde{v}$ we obtain

$$L_m U + \langle F, DU\rangle - \langle \tilde{u}, DU\rangle \leq 0$$

by the definition of H and Eq. (6.3) which is satisfied by both w and $\tilde{v}$. Now, taking the sequence $w_n \to w$ from the first part of this proof and an analogous sequence $(\tilde{v}_n) \subset \operatorname{dom}(L_0)$, $v_n \to v$ we get

$$L_m(w_n - \tilde{v}_n) + \big\langle F, D(w_n - \tilde{v}_n)\big\rangle - \big\langle \tilde{u}, D(w_n - \tilde{v}_n)\big\rangle \leq \delta_n,$$

where $\delta_n \to 0$ uniformly on compacts and δ_n is uniformly polynomially bounded. By the Ito formula we find that

$$E_{x,\tilde{u}} U(X_t) = \lim_{n\to\infty} E_{x.\tilde{u}}\big(w_n(X_t) - \tilde{v}(X_t)\big) \leq \lim_{n\to\infty} E_{x,\tilde{u}} \int_0^t \delta_n(X_s)\, ds \leq 0$$

and therefore by (6.30)

$$U(X_t) = 0 \qquad P_{x,\tilde{u}} = \text{a.s.} \tag{6.31}$$

However, the Markov semigroup corresponding to Eq. (2.2) with any Markov control $\tilde{u} \in \tilde{\mathscr{U}}$ is irreducible (cf. [23]), which means that the closed support of the probability law of X_t is the whole space $\mathscr{X}$ for any $t > 0$. This together with (6.31) and continuity of U implies $U = 0$. ∎

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